Reflection and Transmission at Oblique Angles of incidence 2019

The interface lies at z=0 and is the x-y plane. Here the plane of incidence will lie within the sheet and is parallel to the plane of polarization.

Incident waves:

$$\vec{\tilde{E}}_I=\vec{\tilde{E}}_{0,I}e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}:\vec{\tilde{B}}_I=\vec{\tilde{B}}_{0,I}e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}=\frac{1}{v_1}\left(\hat{k}_I x\vec{\tilde{E}}_I\right)$$

Reflected waves:

$$\vec{\tilde{E}}_R=\vec{\tilde{E}}_{0,R}e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}:\vec{\tilde{B}}_R=\vec{\tilde{B}}_{0,R}e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}=\frac{1}{v_1}\left(\hat{k}_R x\vec{\tilde{E}}_I\right)$$

Transmitted waves:

$$\vec{\tilde{E}}_T=\vec{\tilde{E}}_{0,T}e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}:\vec{\tilde{B}}_T=\vec{\tilde{B}}_{0,T}e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}=\frac{1}{v_2}\left(\hat{k}_T x\vec{\tilde{E}}_T\right)$$

Wave number relations:

$$k_I v_1=k_R v_1=k_T v_2\Rightarrow k_R=k_I=\frac{v_2}{v_1}k_T=\frac{n_1}{n_2}k_T$$

Why? Since the frequencies are all the same, $v=\frac{\omega}{k}\Rightarrow kv=\omega:n=\frac{c}{v}$.

At the interface, for all times, the field on the left side (incident side) must sum to equal the fields on the right side (transmitted side). This means:

$$\vec{k}_I\cdot\vec{r}=\vec{k}_R\cdot\vec{r}=\vec{k}_T\cdot\vec{r}\quad \text{(at z=0).}$$

In terms of the components:

$$k_{I,x}x=k_{R,x}x=k_{T,x}x:k_{I,y}y=k_{R,y}y=k_{T,y}y\Rightarrow k_{I,x}=k_{R,x}=k_{T,x}:k_{I,y}=k_{R,y}=k_{T,y}$$

Note: k may have a z-component but at the interface, z=0.

We are free to orient the x-z plane so that $\hat{k}_I$ lies in the x-z plane and thus the other two vectors will also lie in this plane. However note that this plane is not necessarily parallel to the interface. **This plane is called the “plane of incidence.”**

Since the components of k are equal we then have:

$$k_I\sin\theta_I=k_R\sin\theta_R=k_T\sin\theta_T$$

And the angles are angles of (incidence, reflection, and refraction) respectively.

The law of reflection says: $\theta_I=\theta_R$

Snell's law also applies which says:

$$\frac{\sin\theta_I}{\sin\theta_T}=\frac{n_2}{n_1}$$

The boundary conditions to be satisfied are:

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \qquad E_1^{\parallel} = E_2^{\parallel}$$
$$B_1^{\perp} = B_2^{\perp} \qquad \frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}$$

Here this means:

$$\epsilon_1 \left[\vec{\tilde{E}}_{0,I} + \vec{\tilde{E}}_{0,R} \right]_{x,y} = \epsilon_2 \left[\vec{\tilde{E}}_{0,T} \right]_{x,y}$$
$$\epsilon_1 \left[\vec{\tilde{E}}_{0,I} + \vec{\tilde{E}}_{0,R} \right]_{x,y} = \epsilon_2 \left[\vec{\tilde{E}}_{0,T} \right]_{x,y}$$

$$1: \quad \epsilon_1 \left[\vec{\tilde{E}}_{0,I} + \vec{\tilde{E}}_{0,R} \right]_z = \epsilon_2 \left[\vec{\tilde{E}}_{0,T} \right]_z \quad 2: \quad \left[\vec{\tilde{B}}_{0,I} + \vec{\tilde{B}}_{0,R} \right]_z = \left[\vec{\tilde{B}}_{0,T} \right]_z$$
$$3: \quad \left[\vec{\tilde{E}}_{0,I} + \vec{\tilde{E}}_{0,R} \right]_{x,y} = \left[\vec{\tilde{E}}_{0,T} \right]_{x,y} \quad 4: \quad \frac{1}{\mu_1} \left[\vec{\tilde{B}}_{0,I} + \vec{\tilde{B}}_{0,R} \right]_{x,y} = \frac{1}{\mu_2} \left[\vec{\tilde{B}}_{0,T} \right]_{x,y}$$

Assume the plane of the polarization is parallel to the plane of incidence.

It is worthwhile to show that the transmitted and reflected polarizations are all in the same plane as the incident polarization. (problem 9.14)

Below is this demonstration.

Assume:

$$\hat{n}_I = \hat{x} : \hat{n}_R = \cos\phi_R \hat{x} + \sin\phi_R \hat{y} : \hat{n}_T = \cos\phi_T \hat{x} + \sin\phi_T \hat{y}$$

where the angles are polarization angles with respect to the x axis which is the incident direction of polarization. The three unit vectors show here ,

$$\hat{n}_I, \hat{n}_R, \hat{n}_T \quad ,$$

point along the directions of polarization and are not the indices of refraction.

This means the incident electric field amplitude is, without loss of generality, along the x direction and this only has an x-component (because I said assume that the incident wave is polarized along the x-direction). Looking at boundary condition 3 then shows:

$$\left[\vec{\tilde{E}}_{0,I} + \vec{\tilde{E}}_{0,R} \right]_x = \left[\vec{\tilde{E}}_{0,T} \right]_x$$
$$\left[\vec{\tilde{E}}_{0,R} \right]_y = \left[\vec{\tilde{E}}_{0,T} \right]_y$$

now it is absolutely straight-forward to see that the R and T polarization in the y direction must be at least the same from the second BC. This means $\phi_R = \phi_T$. However, from the first equation, we must have that if this is anything other than zero, there is a component of the electric field which can not be covered by the boundary condition. **This means that the incident, the reflected and the transmitted waves all lie in the same plane of polarization** (but do not all necessarily have the same polarization) so long as the plane of polarization is parallel to the plane of incidence.

....

There are **3 important laws for describing** reflection and transmission:

(1) The plane of incidence is defined by that plane in which the incident, the reflected and the transmitted wave all lie. (They all lie within the plane of incidence). We've decided this means they are in the plane of the sheet (x-z).

(2) The law of reflection : $\theta_I = \theta_R$.

(3) Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

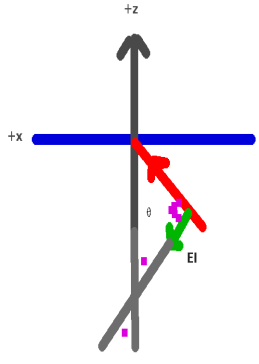
Now let's see what we can do with the boundary conditions (getting rid of the exponentials because they cancel) ... (z=0 at the interface and frequency is the same for all three waves).

$$1: \quad \epsilon_1\left[\tilde{\vec{E}}_{0,I} + \tilde{\vec{E}}_{0,R}\right]_z = \epsilon_2\left[\tilde{\vec{E}}_{0,T}\right]_z \qquad 2: \quad \left[\tilde{\vec{B}}_{0,I} + \tilde{\vec{B}}_{0,R}\right]_z = \left[\tilde{\vec{B}}_{0,T}\right]_z$$

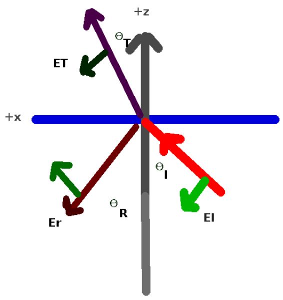
$$3: \quad \left[\tilde{\vec{E}}_{0,I} + \tilde{\vec{E}}_{0,R}\right]_{x,y} = \left[\tilde{\vec{E}}_{0,T}\right]_{x,y} \qquad 4: \quad \frac{1}{\mu_1}\left[\tilde{\vec{B}}_{0,I} + \tilde{\vec{B}}_{0,R}\right]_{x,y} = \frac{1}{\mu_2}\left[\tilde{\vec{B}}_{0,T}\right]_{x,y}$$

Now suppose the polarization of the incident wave is parallel to the plane of incidence which basically means that the electric field amplitude lies within this plane. Note that if this is the case, then the magnetic field amplitudes come out of the sheet or go into the sheet. Then the reflected and transmitted waves have polarizations that also lie in this plane. This is assuming we orient the polarization relative to the electric fields.

This is not going to be the easiest thing to get through but: the incident magnetic field comes in with an angle of 90 degrees with respect to the direction of propagation. This is required for TEM waves. The magnetic field in this instance is assumed and required to lie in the plane of the interface (not the plane of incidence) which thus is used to define the plane or polarization. However, the plane of polarization does not lie in the plane formed by E and B. The normal to this (ExB) plane (which is the direction of propagation given by the Poynting vector) makes the angle θ_I with respect to the normal to the plane of the interface. Also notice that things have been arranged so that the incident electric field intensity is in the x-z plane so that there is no y-component of the incident, reflected or transmitted waves.



The incoming electric field has the orientation shown and lies in the plane of the paper. The arrow represents the direction of propagation . The angle it makes with respect to the normal to the plane of interface is given by $\cos(\theta_I)$. So the z-component of the incident electric field is given by $-\tilde{E}_{0,I}\sin(\theta_I)$ .Note that we have arranged the incident wave so that there is no magnetic z-component associated with the incident, reflected or transmitted waves. The magnetic field points into or out of the paper.



The entire incidence, reflection, and transmission situation looks like this. The magnetic field is at right angles to each of the electric field vectors and is coming out of the page or going into the page for each of the 3 vectors. Thus there is no z-component of the magnetic field. Follow through with the construction that I indicated for the incident electric field and you'll find that the reflected z component of the electric field is given by: $\tilde{E}_{0,R}\cos(\theta_R)$. Looking at E_t you can see that the z-component of this electric field is given by $-\tilde{E}_{0,T}\sin(\theta_T)$. Understanding why the transmitted electric field is as indicated is easy. I think the place where thought is needed is the reflected electric field. A quick (and not completely correct) way to imagine it is this: the E field hits the interface first with the end without the point. That is the first reflected. The arrow end is the last to strike but the other end has already moved away. As I said, not completely correct.

We can now write the results of BC(1):

$$\epsilon_1\left[-\tilde{E}_{0,I}\sin(\theta_I)+\tilde{E}_{0,R}\sin(\theta_R)\right]=\epsilon_2\left[-\tilde{E}_{0,t}\sin(\theta_T)\right]$$

I think that possibly the easiest way to get the components here is to extend them towards the optical axis and use knowledge of the right triangles that form to get this. We have in this instance required the magnetic fields to lie in the plane of incidence so the second boundary condition give no new information.

The third boundary condition gives (through the same process I showed earlier):

$$\left[\tilde{E}_{0,I}\cos(\theta_I)+\tilde{E}_{0,R}\cos(\theta_R)\right]=\tilde{E}_{0,T}\cos(\theta_T)$$

Now remember that the magnetic fields are related to the electric fields by

$$\tilde{B}_{0,T}=\frac{1}{v}\hat{k}\times\vec{E}$$

let's look at the 4th boundary condition.

$$\frac{1}{\mu_1}\left[\vec{\tilde{B}}_{0,I}+\vec{\tilde{B}}_{0,R}\right]_{x,y}=\frac{1}{\mu_2}\left[\vec{\tilde{B}}_{0,T}\right]_{x,y}$$

In order to evaluate this, we need to work with the components of the electric fields that were obtained above. It kind-of goes like this:

(a) find the components of $\hat{k}_1$: $\hat{k}_1 = \sin(\theta_I)\hat{x} + \cos(\theta_i)\hat{z}$

$\vec{\tilde{E}}_{0,I} = \tilde{E}_{0,I}\cos(\theta_I)\hat{x} - \tilde{E}_{0,I}\sin(\theta_I)\hat{z}$

so: $\hat{k}_1 \times \vec{\tilde{E}}_{0,I} = \tilde{E}_{0,I}\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin(\theta_I) & \cos(\theta_I) & 0 \\ \cos(\theta_I) & -\sin(\theta_I) & 0 \end{bmatrix} = \tilde{E}_{0,I}\left[0\hat{x} - 0\hat{y} + \left(-\sin^2(\theta_I) - \cos^2(\theta_I)\right)\right] = -\tilde{E}_{0,I}$

(b) find the components of $\hat{k}_2$: $\hat{k}_2 = \sin(\theta_R)\hat{x} - \cos(\theta_R)\hat{z}$

$\vec{\tilde{E}}_{0,R} = \tilde{E}_{0,R}\cos(\theta_R)\hat{x} + \tilde{E}_{0,R}\sin(\theta_R)\hat{z}$

$\hat{k}_2 \times \vec{\tilde{E}}_{0,R} = \tilde{E}_{0,R}\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin(\theta_R) & -\cos(\theta_R) & 0 \\ \cos(\theta_R) & \sin(\theta_R) & 0 \end{bmatrix} = \tilde{E}_{0,R}\left[\sin^2(\theta_R) + \cos^2(\theta_R)\right]\hat{z} = \tilde{E}_{0,R}\hat{z}$

(c) find the components of $\hat{k}_3$: $\hat{k}_3 = \sin(\theta_T)\hat{x} + \cos(\theta_T)\hat{z}$

$\vec{\tilde{E}}_{0,T} = \tilde{E}_{0,T}\cos(\theta_T)\hat{x} - \tilde{E}_{0,T}\sin(\theta_T)\hat{z}$

$\hat{k}_3 \times \vec{\tilde{E}}_{0,T} = \tilde{E}_{0,R}\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin(\theta_T) & \cos(\theta_T) & 0 \\ \cos(\theta_T) & -\sin(\theta_T) & 0 \end{bmatrix} = \tilde{E}_{0,R}\left(-\sin^2(\theta_T) - \cos^2(\theta_T)\right)\hat{z} = -\tilde{E}_{0,R}\hat{z}$

We then have the fourth boundary condition gives:

$$\frac{1}{\mu_1 v_1}\left[-\tilde{E}_{0,I} + \tilde{E}_{0,R}\right] = \frac{1}{\mu_2 v_2}\left(-\tilde{E}_{0,R}\right) \Rightarrow \frac{1}{\mu_1 v_1}\left[\tilde{E}_{0,I} - \tilde{E}_{0,R}\right] = \frac{1}{\mu_2 v_2}\tilde{E}_{0,T} \quad \text{(eq 9.104)}$$

It is easy to see that this condition reduces to:

$$\tilde{E}_{0,I} - \tilde{E}_{0,R} = \frac{\mu_1 v_1}{\mu_2 v_2}\tilde{E}_{0,T} = \beta\tilde{E}_{0,T}$$

Where $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$

With the law of reflection , the first condition is:

$$\epsilon_1\left[-\tilde{E}_{0,I} + \tilde{E}_{0,R}\right]\sin(\theta_I) = \epsilon_2\left[-\tilde{E}_{0,t}\sin(\theta_T)\right] \Rightarrow \epsilon_1\left[\tilde{E}_{0,I} - \tilde{E}_{0,R}\right]\sin(\theta_I) = \epsilon_2\left[\tilde{E}_{0,T}\sin(\theta_T)\right]$$

Now we use Snell's law:, noting that $\frac{\sin(\theta_T)}{\sin(\theta_I)} = \frac{n_1}{n_2}$ gives

$$n_1\sin(\theta_1) = n_2\sin(\theta_2) \Rightarrow \left[\tilde{E}_{0,I} - \tilde{E}_{0,R}\right] = \frac{\epsilon_2}{\epsilon_1}\frac{n_2}{n_1}\tilde{E}_{0,T} \quad \text{using:} \quad \frac{\epsilon_2 n_2}{\epsilon_1 n_1} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} = \frac{\frac{v_2}{v_2^2 \mu}}{\frac{v_1}{v_1^2 \mu_1}} = \frac{\mu_1 v_1}{\mu_2 v_2}$$

gives:

$$\tilde{E}_{0,I} - \tilde{E}_{0,R} = \frac{\mu_1 v_1}{\mu_2 v_2}\tilde{E}_{0,T} = \beta\tilde{E}_{0,T} \quad .$$

Note that the last boundary condition gives no new information.

The remaining boundary condition was: $\left[\tilde{E}_{0,I}\cos(\theta_I)+\tilde{E}_{0,R}\cos(\theta_R)\right]=\tilde{E}_{0,T}\cos(\theta_T)$
Again, using the law of reflection, we have:

$$\tilde{E}_{0,I}+\tilde{E}_{0,R}=\frac{\cos(\theta_T)}{\cos(\theta_I)}\tilde{E}_{0,T}=\alpha\tilde{E}_{0,T} \quad \text{where} \quad \alpha\equiv\frac{\cos(\theta_T)}{\cos(\theta_I)} \ .$$

To recap: we have these two results: $\tilde{E}_{0,I}-\tilde{E}_{0,R}=\beta\tilde{E}_{0,T}$ and $\tilde{E}_{0,I}+\tilde{E}_{0,R}=\alpha\tilde{E}_{0,T}$

If we add these two results: $(\alpha+\beta)\tilde{E}_{0,T}=2\tilde{E}_{0,I}\Rightarrow\tilde{E}_{0,T}=\frac{2}{\alpha+\beta}\tilde{E}_{0,I}$

If we now use this to solve for the reflected field: $\tilde{E}_{0,R}=\left[1-\frac{2\beta}{\alpha+\beta}\right]\tilde{E}_{0,I}=\left[\frac{\alpha-\beta}{\alpha+\beta}\right]\tilde{E}_{0,I}$.

These two equations constitute "Fresnel's Equations." There are 2 more of his equations dealing with the case where the polarization is perpendicular to the interface.
There are several special cases. Suppose $\theta_I=90^o$. Then: $\tilde{E}_{0,R}=\tilde{E}_{0,I}$: the light is completely reflected. Your author reminds you of driving at night on a wet road.

There is also another case: if $\alpha=\beta;\beta\equiv\frac{\mu_1 v_1}{\mu_2 v_2}:\mu_1=\mu_2\Rightarrow\beta=\frac{n_2}{n_1}$ then the reflected intensity is zero. This is Brewster's angle. Let's obtain it.

$$\alpha=\beta\Rightarrow\frac{\cos(\theta_T)}{\cos(\theta_I)}=\beta\Rightarrow 1-\sin^2(\theta_T)=\beta^2\left(1-\sin^2(\theta_I)\right)\Rightarrow 1-\left[\frac{n_1^2\sin^2(\theta_I)}{n_2^2}\right]=\beta^2-\beta^2\sin^2(\theta_I)$$

$$\Rightarrow 1=\beta^2-\beta^2\sin^2(\theta_I)+\frac{n_1^2}{n_2^2}\sin^2(\theta_I)\Rightarrow 1-\beta^2=\sin^2(\theta_I)\left(\frac{n_1^2}{n_2^2}-\beta^2\right)\Rightarrow\sin^2(\theta_I)=\frac{1-\beta^2}{\left(\frac{n_1}{n_2}\right)^2-\beta^2}$$

This is the famous result below:

$$\text{if } \mu_1\approx\mu_2\Rightarrow\beta\approx\frac{n_2}{n_1}\Rightarrow\sin^2(\theta_I)\approx\frac{1-\left(\frac{n_2^2}{n_1^2}\right)}{\frac{n_1^2}{n_2^2}-\frac{n_2^2}{n_1^2}}\Rightarrow\sin\theta_I=\frac{(n_2/n_1)}{\sqrt{1+(n_2/n_1)^2}}\Rightarrow\tan\theta_I\approx\frac{n_2}{n_1}\equiv\tan\theta_B \ .$$

We can also get the reflection and transmission coefficients from this:

$$I_I=\vec{S}\cdot\hat{z}=\frac{1}{2}\epsilon_1 v_1 E_{0,I}^2\cos(\theta_I):R\equiv\frac{I_R}{I_I}=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2:T\equiv\frac{I_T}{I_I}=\alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

Also you should note that if $\alpha>\beta$ then the reflected wave is 180° out of phase with the incident wave. But you expected this also from the problem of the string with the knot.
There is also the critical angle. I refer you to problem 9.37 for work related to the critical angle. (it is well worth reading at the minimum).