Geometrical Optics

These notes show you how the various important equations that we have used are obtained. You will not be expected be able to derive these results on a test.

Derivation of

(1) the definition for magnification

(2) the mirror equation and the focal length of a mirror

(3) the magnification of a refracting surface

(4) the thin lens equation, and

(5) the lens makers' equation for the focal length of a lens.

I: How to do optical ray tracing (for mirrors)

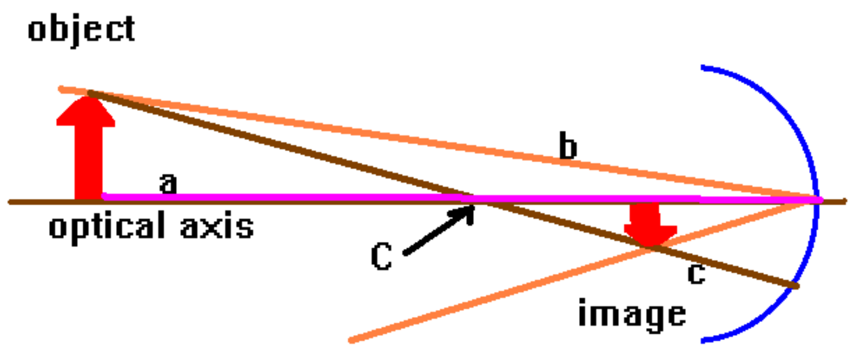
(a) Let the first ray leave the bottom of the object and travel along the optical axis. {the optical axis is the symmetry axis of the mirror}

(b) Let the second ray leave the top of the object and strike the mirror where the optical axis touches the mirror.

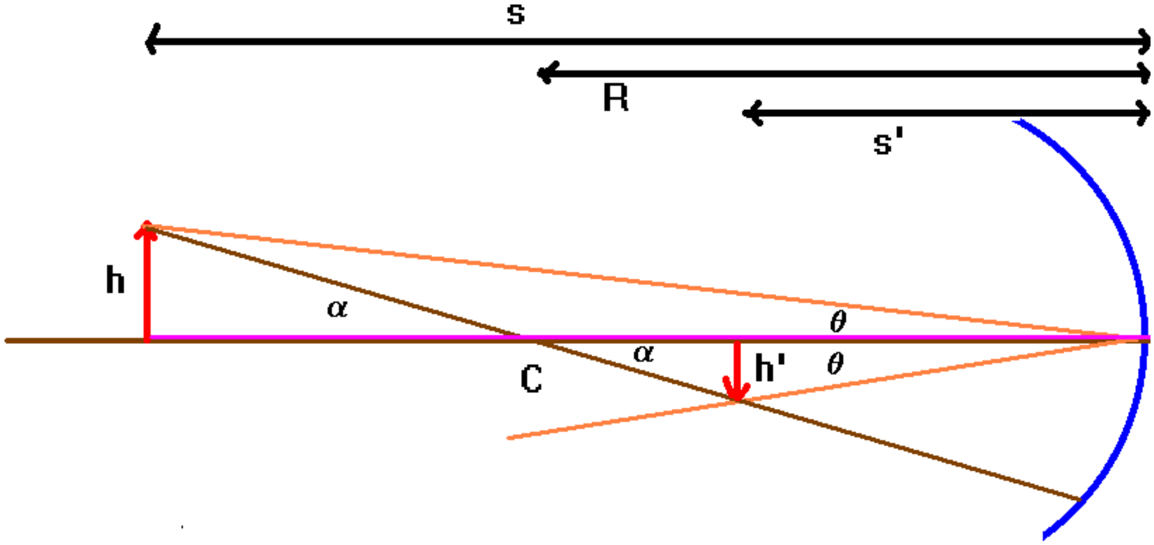
(c) Let the third ray leaves the top of the object and pass through the center of curvature.

Of these three rays, only the second ray will not be reflected back upon itself.

The image will form at the intersection of rays b and c as shown.



II: How does ray tracing lead to the mirror equation?



First let's show that

$$M \equiv \frac{h'}{h} = \frac{-s'}{s}$$

Look at the angles marked θ in the figure above.
The law of reflection says that the two angles are the same. Then:

$$\tan(\theta) = \frac{h}{s} \text{ and } \tan(\theta) = -\frac{h'}{s'}$$

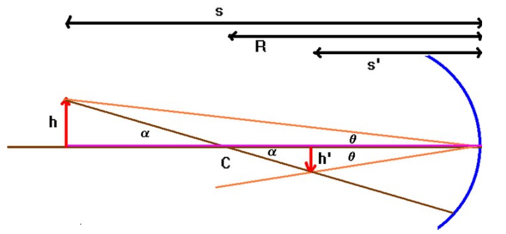
So we then have the result

$$\frac{h}{s} = -\frac{h'}{s'}$$

We see then that

$$\frac{h'}{h} = -\frac{s'}{s} \equiv M$$

Now let me show you where the mirror equation comes from.



$$\tan(\alpha)=\frac{h}{s-R} \quad \text{and} \quad \tan(\alpha)=-\frac{h'}{R-s'}$$

Divide these two equations:

$$\frac{\tan(\alpha)}{\tan(\alpha)}=1=\frac{\frac{h}{s-R}}{-\frac{h'}{R-s'}}=-\frac{h}{h'}\frac{R-s'}{s-R}=\frac{s(R-s')}{s'(s-R)}$$

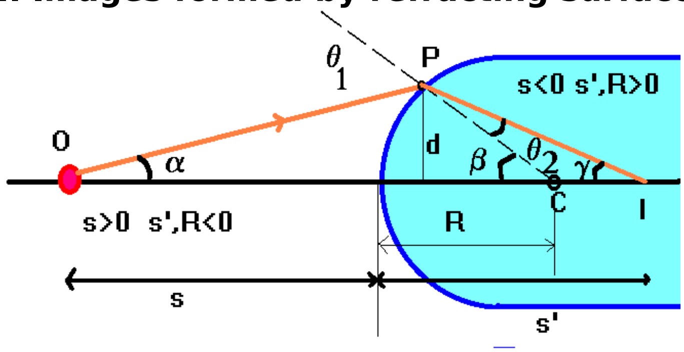
We can show that this reduces to the mirror equation as follows:

$$1=\frac{s(R-s')}{s'(s-R)}\Rightarrow s's-s'R=sR-ss'\Rightarrow 2ss'=(s+s')R\Rightarrow\frac{ss'}{s+s'}=\frac{R}{s}\Rightarrow\frac{s+s'}{ss'}=\frac{2}{R}=\frac{1}{s}+\frac{1}{s'}$$

If we define the focal length as R/2 then the mirror equation results.

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f} \text{ where } f=\frac{R}{2}$$

III. Images formed by refracting surfaces



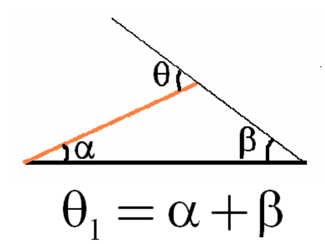
Snell's law tells us how θ_1 and θ_2 are related: $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$.

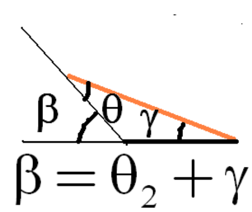
Paraxial approximation: the rays are parallel to the optical axis and not far from it.

If you assume paraxial rays, then Snell's law is approximately given by

$$n_1 \theta_1 = n_2 \theta_2$$

Geometry=> "exterior angle of any triangle =the sum of the two opposite interior angles"





$\theta_1 = \alpha + \beta$ and $\beta = \theta_2 + \gamma$

combine this result with the approximated form of Snell's law:

$$n_1(\alpha + \beta) = n_2(\beta - \gamma) \Rightarrow n_1 \alpha + n_1 \beta = n_2 \beta - n_2 \gamma$$

This simplifies to: $n_1 \alpha + n_2 \gamma = (n_2 - n_1)\beta$

Trigonometry (and small angles) =>

$$\tan\alpha = \frac{d}{s} \approx \alpha : \tan\beta = \frac{d}{R} \approx \beta : \tan\gamma = \frac{d}{s'} \approx \gamma$$

Use these results in the *simplified* Snell's law:

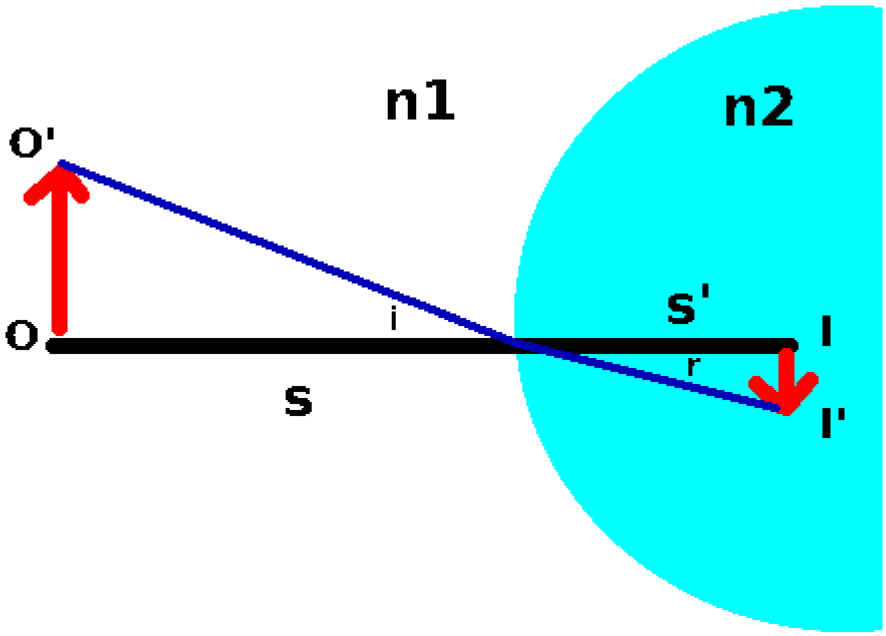
$$\frac{n_1 d}{s} + \frac{n_2 d}{s'} = \frac{d(n_2 - n_1)}{R}$$

One more simplification: divide by d to get:

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{(n_2 - n_1)}{R}$$

This equation above tells us where the image forms for a single refracting surface.

Magnification of image formed from a spherical surface



Looking at the image above,

$$\tan(i)=\frac{\overline{oo'}}{s}\approx i : \tan(r)=-\frac{\overline{II'}}{s'}\approx r$$

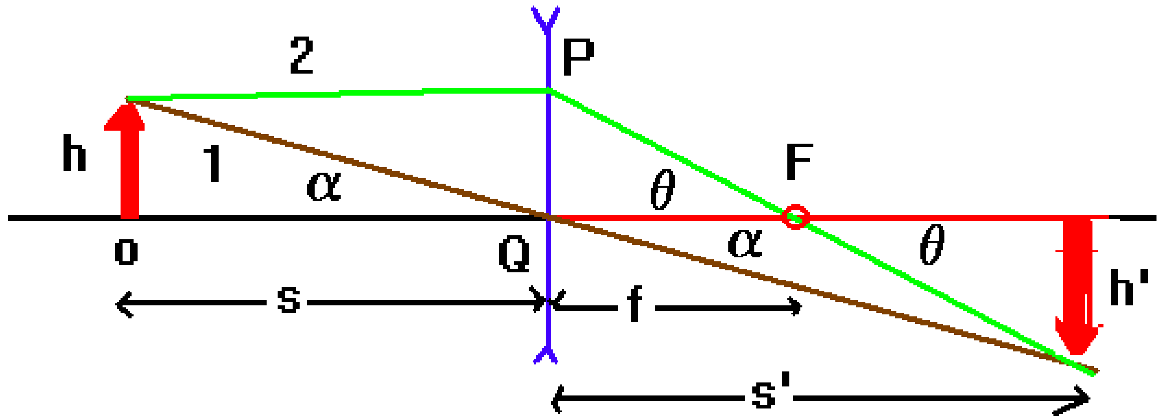
$$n_1 \sin(i)=n_2 \sin(r)\Rightarrow n_1 i\approx n_2 r$$

$$\Rightarrow n_1 \frac{\overline{oo}}{s}=-n_2 \frac{\overline{II'}}{s'}\Rightarrow \frac{\overline{II'}}{\overline{OO'}}=-\frac{n_1 s'}{n_2 s}=M$$

The magnification for a refracting surface such as this ***is different*** from what we have had before. It is given by

$$M=-\frac{n_1 s'}{n_2 s}$$

IV. **The thin lens equation**



Note the rays marked 1 and 2 above.

This also shows you how to do raytracing with lenses. ray 1 passes through the vertex of the lens and is not deviated. ray 2 is parallel to the optical axis and thus will pass through the focal point of the lens. Intersect these two rays (or their extensions) to get the image location.

$\tan\alpha=\frac{h}{s}$ and $\tan\alpha=-\frac{h'}{s'}$. Thus $M\equiv\frac{h'}{h}=-\frac{s'}{s}$ as with mirrors.

Look at the angles marked θ (they are both the same because of intersecting lines)

$$\tan\theta=\frac{\overline{PQ}}{f}=\frac{h}{f} \quad \text{and} \quad \tan\theta=-\frac{h'}{s'-f}$$

thus

$$\frac{h}{f}=-\frac{h'}{s'-f}\Rightarrow\frac{h'}{h}=\frac{s'-f}{f}=\frac{s'}{s} \quad \text{(using the result for M above)}$$

Let's cast this into the form of the thin lens equation by dividing by s':

$$\frac{\left(\frac{s'-f}{s'}\right)}{f}=\frac{\left(\frac{s'}{s'}\right)}{s}\Rightarrow\frac{\left(1-\frac{f}{s'}\right)}{f}=\frac{1}{s}\Rightarrow\frac{1}{f}-\frac{1}{s'}=\frac{1}{s}$$

Thus you see the thin lens equation results:

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

V. **Where does the lensmaker's equation come from?**

Recall how images behave at refracting surfaces:

$$\frac{n_1}{s_1}+\frac{n_2}{s'_1}=\frac{(n_2-n_1)}{R_1}\Rightarrow\frac{n_2}{s'_1}=\frac{(n_2-n_1)}{R_1}-\frac{n_1}{s_1}$$

The original object is at s_1 and the first image is at s_1’. Imagine a thin lens surrounded by a medium of index of refraction n_1. Now let's form a second surface and look for the image:

$$\frac{n_2}{s_2}+\frac{n_1}{s'_2}=\frac{(n_1-n_2)}{R_2}=-\frac{(n_2-n_1)}{R_2}$$

The (second) object is at s_2 and the (second) image is at s_2’.

Since the lens is thin, the image from the first side forms the object for the second side but it is virtual. Thus $s_2=-s'_1$ ***if the lens is not thin, this needs to be modified***.

I use this in the second equation to give:

$$\frac{n_2}{s_2}+\frac{n_1}{s'_2}=-\frac{(n_2-n_1)}{R_2}\Rightarrow-\frac{n_2}{s'_1}+\frac{n_2}{s'_2}=\frac{-(n_2-n_1)}{R_2}$$

Now I use the results from the first equation and obtain:

$$\frac{n_1}{s_1}+\frac{n_2}{s'_1}=\frac{(n_2-n_1)}{R_1}\Rightarrow\frac{n_2}{s'_1}=\frac{(n_2-n_1)}{R_1}-\frac{n_1}{s_1}$$

$$-\frac{n_2}{s'_1}+\frac{n1}{s'_2}=\frac{-(n_2-n_1)}{R_2}\Rightarrow-\left[\frac{(n_2-n_1)}{R_1}-\frac{n_1}{s_1}\right]+\frac{n_1}{s'_2}=-\frac{(n_2-n_1)}{R_2}$$

A little bit of algebra:

$$\frac{n_1}{s_1}+\frac{n_1}{s'_2}=-\frac{(n_2-n_1)}{R_2}+\left[\frac{(n_2-n_1)}{R_1}\right]=(n_2-n_1)\left[\frac{1}{R_1}-\frac{1}{R_2}\right]$$

Divide by n_1 to obtain:

$$\frac{n_1}{s_1}+\frac{n_1}{s'_2}=(n_2-n_1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\Rightarrow\frac{1}{s}+\frac{1}{s'_2}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

Then recognize that s_1=s and $s’_2$=s’ that we used in the thin lens equation. Then we have:

$$\frac{1}{s}+\frac{1}{s'}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

If the lens is in air, then $n_1=1$ so the lensmaker's equation becomes

$$\frac{1}{s}+\frac{1}{s'}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

You can't help but notice that this is the focal length in the thin lens equation. Thus

$$\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

and the thin lens equation is then: $\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$.