Relativity of simultaneity and preservation of causality

Start with the Lorentz transformations:

$$x'=\gamma(x-vt) \qquad y'=y \qquad z'=z \qquad t'=\left[\frac{t-\frac{vx}{c^2}}{\sqrt{1-\beta^2}}\right]$$

or the inverse transformations:

$$x=\gamma(x'+vt') \qquad y=y' \qquad z=z' \qquad t=\left[\frac{t'+\frac{vx}{c^2}}{\sqrt{1-\beta^2}}\right]$$

.

Frank (the unprimed frame) sees Mary move with a velocity v. Let the coordinate be synchronized at t=0,x=0. Consider two events in Frank's Frame, which are simultaneous in Frank's frame but occur at different x-values. Let the first event be at x_1 and the second event be at x_2. Now transform to Mary's frame.

$$t_1'=\gamma\left[t-\frac{v\,x_1}{c^2}\right]:t_2'=\gamma\left[t-\frac{v\,x_2}{c^2}\right] \quad .$$

Here, t is the common time at which the simultaneous events are observed in Frank's frame (and thus requires no subscript). In Mary's frame, however the events are not simultaneous: the are temporally separated by:

$$\delta t\equiv t_2'-t_1'=-\gamma\left[\frac{v\,x_2}{c^2}-\frac{v\,x_1}{c^2}\right]=\gamma\frac{v}{c^2}(x_1-x_2) \quad .$$

Only in the event that the two events also occur at the same spatial coordinates in Frank's frame will the events be simultaneous in Mary's frame. If this were the case, however, the two events would be the same event. Thus we have the conclusion that simultaneity in one frame does not imply simultaneity in all frames.

What then about causality?

Suppose in Frank's frame, in the simplest situation, two events separated by $\delta t=t_2-t_1$ occur at the same x-values. Since this happens in Frank's frame, this will be the shortest time increment between the two events. Never-the-less, let us see what happens in Mary's frame.

$$t_1'=\gamma\left[t_1-\frac{v\,x}{c^2}\right]:t_2'=\gamma\left[t_2-\frac{v\,x}{c^2}\right]:\delta t'\equiv t_2'-t_1'=\gamma(t_2-t_1)=\gamma\,\delta t$$

In particular, suppose event 1 caused event 2. Then we see the two events will retain causality in the transformed frame.

Now going just a little bit further, let the two events be also spatially separated. Then:

$$t_1'=\gamma\left[t_1-\frac{v\,x_1}{c^2}\right]:t_2'=\gamma\left[t_2-\frac{v\,x_2}{c^2}\right]:\delta t'\equiv t_2'-t_1'=\gamma\left[\delta t-\frac{v}{c^2}\delta x\right]$$

In order for the two events to be simultaneous in Mary's frame, we then require:

$$\delta t=\frac{v}{c^2}\delta x\Rightarrow\delta x=\frac{c^2}{v}\delta t$$

If the two events are connected by a light signal, this gives the minimum time increment between the two events so let's look at this situation:

Suppose: $\frac{\delta x}{\delta t}=c$. then $\delta x=c\delta t \Rightarrow \delta t'=\gamma\left[\delta t-\frac{v}{c^2}(c\delta t)\right]=\gamma\delta t\left[1-\frac{v}{c}\right]$. For the events to be simultaneous in Mary's frame (and therefore not causal) we have the requirement that Mary's frame move at the speed of light, which is not possible. Furthermore since the connection by a light signal in Frank's frame is the boundary of causality (the actual time increment will be larger for causality) this means that Mary can not observe two events as being simultaneous in her frame if they were causal in Frank's frame.