Relativistic Doppler Shift r19
Spaceship (source) moving directly away from the earth (receiver)

Suppose, that an observer in the earth's frame of reference is receiving a light pulse is sent out every ct , from a space ship. A spaceship moves away from the earth with a speed βc .

How often does the earth receive signals?

Unlike the classical Doppler shift, we can not change the speed of light since everyone observes the same value here. So the spaceship sent out a pulse at t_1 and a second pulse at t_2. Since this is in the rest frame of the spaceship, this would be the proper time interval.

$$\delta t_{source} = t_{2,source} - t_{1,source} \quad .$$

However, according to the spaceship and the earth-bound observer, the earth has moved further away from the spaceship. The earth coordinates are x_1' and x_2' . Effectively the wavelength observed on the earth will then be longer by the amount $\beta c t$ .But since everyone observed the same speed of light, we must have:

$$\lambda_{source} + \beta c t = ct \Rightarrow t = \frac{\lambda_{source}}{c(1-\beta)} = \frac{c}{f_{source} c(1-\beta)} = \frac{1}{(1-\beta) f_{source}} \Rightarrow f_{observer} = (1-\beta) f_{source} \quad .$$

Note that this f_{source} would be the frequency of the source without time dilation taken into account. Again, using the Lorentz transformations for time we get the time dilation factor:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left[t - \frac{vx}{c^2}\right] \end{aligned} \quad \text{or the inverse transformations:} \quad \begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left[t' + \frac{vx}{c^2}\right] \end{aligned} \quad .$$

$$\delta t' = t_2' - t_1' = \gamma\left[t_2 - \frac{v x_2}{c^2}\right] - \gamma\left[t_1 - \frac{v x_1}{c^2}\right] = \gamma\left[\delta t - \frac{v(x_1 - x_2)}{c^2}\right] = \gamma[\delta t - \beta^2 \delta t] = \frac{\delta t}{\gamma} \quad .$$

This means that the periods will transform as:

$$\delta t = \gamma \delta t' \Rightarrow f_{source} = \frac{f_{observer}}{\gamma} \Rightarrow f_{observer} = \gamma f_{source}$$

(according to the source, the earth is moving).

Ok, put it all together now and obtain:

$$\gamma(1-\beta) = \frac{1-\beta}{\sqrt{1-\beta^2}} = \frac{\sqrt{(1-\beta)^2}}{\sqrt{(1-\beta)(1+\beta)}} = \sqrt{\frac{1-\beta}{1+\beta}} \Rightarrow f_{observer} = \sqrt{\frac{1-\beta}{1+\beta}} f_{source}$$

If instead the spaceship is moving towards the earth, then change the sign of the velocity above to obtain:

$$f_{observer} = \sqrt{\frac{1+\beta}{1-\beta}} f_{source}$$

Note that you can write this in terms of wavelength to get the "redshift" or the blue shift. The blue shift will be seen when the spaceship moves towards the earth, the redshift will be seen when it moves away from the earth.