| r22 | Physics**220**: Worksheet 03 | Name _____ |

Concepts: Gauss's Law, Electric Flux, superposition of electric fields.

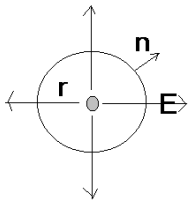Note: $\epsilon_0 = 8.854 \times 10^{-12} \dfrac{C^2}{Nm^2}$

(1) A point charge +q is located at the origin. Use Gauss's law to find $\vec{E}$ everywhere.

(2) A sphere of radius a has a total charge +Q uniformly distributed over its volume. Use Gauss's law to find $\vec{E}$ inside the sphere and outside the sphere.

(3) An infinitely long wire has a charge per unit length +λ. Use Gauss's law to find $\vec{E}$ everywhere.

(4) An infinite plane has a surface charge density σ. Use Gauss's law to find $\vec{E}$ everywhere.

(5) Two infinite planes are separated by a distance d. Plane 1 has a surface charge +σ while plane 2 has a surface charge -σ. Use Gauss's law to find $\vec{E}$ everywhere.

(1) A point charge +q is located at the origin. Use Gauss's law to find $\vec{E}$ everywhere.

(a) Choose the Gaussian surface: Concentric sphere centered on charge of radius r centered on the charge distribution. On this surface, $\vec{E}$ is parallel to $\vec{n}$ and constant in magnitude: that is actually why we chose this particular geometry.

(b) Calculate the electric flux through the Gaussian surface:
$$\Phi_E = \sum_{\Delta A_i} \vec{E}_i \cdot \Delta \vec{A}_i = \sum_{\Delta A_i} \vec{E}_i \cdot \hat{n} \Delta A_i = \sum_{\Delta A_i} E_i \cdot \Delta A_i = E \sum_{\Delta A_i} A_i = E(4\pi r^2)$$

(c) Calculate the net charge enclosed by your Gaussian Surface:
$$Q_{enc} = \sum_{\substack{volume\,of \\ Gaussian\,Surface}} \rho(\vec{r}) \Delta V_i = q$$

(d) Use expressions from (b) and (c) in Gauss's law.

By Gauss's law: $\Phi_E = \dfrac{Q_{enc}}{\epsilon_0} \Rightarrow E(4\pi r^2) = \dfrac{q}{\epsilon_0}$

(e) Solve the result in (d) for the electric field.
$$E = \frac{q}{4\pi\epsilon_0 r^2}$$

(f) Apply your argument for the direction of the electric field in (a) to provide the vector electric field.
$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

Note from Coulomb's law, since $k = \dfrac{1}{4\pi\epsilon_0}$, this becomes $\vec{E} = k\dfrac{q}{r^2}\hat{r}$.

(2) A sphere of radius a has a total charge +Q uniformly distributed over its volume. Use Gauss's law to find $\vec{E}$ inside the sphere and outside the sphere.

(a) Choose the Gaussian surface: Concentric sphere centered on charge of radius r. On this surface, $\vec{E}$ is parallel to $\hat{n}$ and constant in magnitude: that is actually why we chose this particular geometry. The radius of this sphere may be either greater than a or less than a.

Outside:

(b) Calculate the electric flux through the Gaussian surface:

$$\Phi_E = \sum_{\Delta A_i} \vec{E}_i \cdot \Delta \vec{A}_i = \sum_{\Delta A_i} \vec{E}_i \cdot \hat{n} \Delta A_i = \sum_{\Delta A_i} E_i \cdot \Delta A_i = E \sum_{\Delta A_i} A_i = E(4\pi r^2)$$

(c) Calculate the net charge enclosed by your Gaussian Surface:

$$Q_{enc} = \sum_{\substack{\text{volume of} \\ \text{Gaussian Surface}}} \rho(\vec{r}) \Delta V_i = \rho \sum_{\substack{\text{volume of} \\ \text{Gaussian Surface}}} \Delta V_i = \rho\left(\frac{4}{3}\pi r^4\right) = \frac{Q}{\left(\frac{4}{3}\pi a^3\right)}\left(\frac{4}{3}\pi r^3\right) = Q\left(\frac{r}{a}\right)^3$$

Note: outside the Gaussian surface encloses the entire charge. However, you probably want to see how to determine the volume charge density, $\rho(\vec{r})$.

We are given that the charge is **uniformly distributed** over the volume of the sphere. This then implies that the volume charge density is constant. We can find this density easily then:

$$\rho(\vec{r}) = \frac{Q}{\text{Volume}_{sphere}} = \frac{Q}{\left(\frac{4}{3}\pi a^3\right)}$$

You can make this more complete. Since the sphere has a radius a, at regions outside the sphere, there is no additional charge. Thus, in fact, you probably want to write this volume charge density as:

$$\rho(\vec{r}) = \frac{Q}{\text{Volume}_{sphere}} = \frac{Q}{\left(\frac{4}{3}\pi a^3\right)}[r \leq a]$$

$$0[r > a]$$

$$\text{inside} : Q_{enc} = Q\left(\frac{r}{a}\right)^3 : \text{Outside} : Q_{enc} = Q$$

(d) Use expressions from (b) and (c) in Gauss's law and solve for $|\vec{E}|$:
(i): Outside the sphere:

$$E\left(4\pi r^2\right)=\frac{Q}{\epsilon_0}\Rightarrow|\vec{E}|=\frac{Q}{4\pi\epsilon_0 r^2}$$

(ii): Inside the sphere:

$$E\left(4\pi r^2\right)=\frac{Q}{\epsilon_0}\left(\frac{r}{a}\right)^3\Rightarrow|\vec{E}|=\frac{Q}{4\pi\epsilon_0}\frac{r}{a^3}$$

(f) Apply your argument for the direction of the electric field in (a) to provide the vector electric field.
Since E is in the radial direction, we then have:

(i): Outside the sphere:

$$\vec{E}=\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

(ii): Inside the sphere:

$$\vec{E}=\frac{Q}{4\pi\epsilon_0}\frac{r}{a^3}\hat{r}$$

Notice that at r=a, both solutions are the same.

Here is an important thing to note from this problem: Your Gaussian surface that you chose is the thing that determines how much charge is enclosed. In particular, if (as in this case) the charge density varied with radius, and your Gaussian surface extends to a radius r , you only enclose the charge which is inside the surface.

(3) An infinitely long wire has a charge per unit length $+\lambda$. Use Gauss's law to find $\vec{E}$ everywhere.
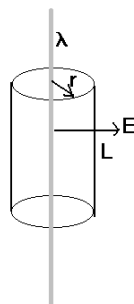
(a) Choose the Gaussian surface: Cylinder of height h, radius s, with the cylinder axis concentric with the wire. On this surface, $\vec{E}$ is parallel to $\hat{n}$ and constant in magnitude: that is actually why we chose this particular geometry. On the ends, however, E and the normal would be perpendicular so no net flux passes through the ends in the ideal situation. Note that on the lateral surface, the cylindrical unit vector $\hat{s}$ and the normal $\hat{n}$ satisfy $\hat{s} \| \hat{n}$.

(b) Calculate the electric flux through the Gaussian surface: at a point on the Gaussian surface: Gauss's law: $\Phi_E = \dfrac{Q_{enc}}{\epsilon_0}$. So

$$\Phi_E = \sum_{\Delta A_i} \vec{E}_i \cdot \Delta \vec{A}_i = \sum_{\Delta A_i} \vec{E}_i \cdot \hat{n} \Delta A_i = \sum_{\Delta A_i} E_i \cdot \Delta A_i = E \sum_{\Delta A_i} A_i = E(2\pi sh)$$

(c) Calculate the net charge enclosed by your Gaussian Surface:

$$Q_{enc} = \sum_{\substack{volume\,of \\ Gaussian\,Surface}} \rho(\vec{r}) \Delta V_i = \lambda(h)$$

Here, we will assume the wire has zero thickness. Then any Gaussian surface we choose except for one which has zero surface area encloses a net charge which is given then by:

$$Q_{enc} = \lambda h$$

(d) Use expressions from (b) and (c) in Gauss's law. Gauss's law: $\Phi_E = \dfrac{Q_{enc}}{\epsilon_0}$, so

$$\Phi_E = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E(2\pi sh) = \frac{\lambda h}{\epsilon_0}$$

(e) Solve the result in (d) for the electric field.

$$E = \frac{\lambda h}{2\pi sh\epsilon_0} = \frac{\lambda}{2\pi \epsilon_0 s}$$

(f) Apply your argument for the direction of the electric field in (a) to provide the vector electric field.

In this problem, the electric field always points away from the wire which is also the cylindrical coordinate $\hat{s}$ direction. The electric field is thus given by:

$$\vec{E} = \frac{\lambda}{2\pi \epsilon_0 s} \hat{s}$$

(4) An infinite plane has a surface charge density σ. Use Gauss's law to find $\vec{E}$ everywhere.
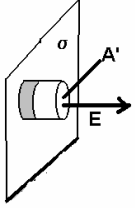
(a) Choose the Gaussian surface: Use a Gaussian surface which is a cylinder cutting through the plane with a face which has an area A'. The electric field only passes through A' and is perpendicular to the sided. We'll call this the $\hat{x}$ direction.

(b) Calculate the electric flux through the Gaussian surface:

$$\Phi_{E,+end}=\sum_{\Delta A_i}\vec{E}_i\cdot\Delta\vec{A}_i=\sum_{\Delta A_i}\vec{E}_i\cdot\hat{n}\Delta A_i=\sum_{\Delta A_i}E_i\cdot\Delta A_i=E\sum_{\Delta A_i}A_i=EA'$$

$$\Phi_{E,-end}=\sum_{\Delta A_i}\vec{E}_i\cdot\Delta\vec{A}_i=\sum_{\Delta A_i}\vec{E}_i\cdot\hat{n}\Delta A_i=\sum_{\Delta A_i}E_i\cdot\Delta A_i=E\sum_{\Delta A_i}A_i=EA'$$

Note that the cylinder has two faces. In the $-\hat{x}$ region of space, the electric field also points in the direction of the normal on the end-piece. This gives the total flux as:
$$\Phi_E=2EA'$$
On the sides of the cylinder, there is no contribution to the flux since the electric field and the normal are perpendicular to each other.

(c) Calculate the net charge enclosed by your Gaussian Surface:
$$Q_{enc}=\sum_{\substack{\text{volume of}\\\text{Gaussian Surface}}}\rho(\vec{r})\Delta V_i=\sigma(A')$$

(d) Use expressions from (b) and (c) in Gauss's law.

By Gauss's law: $\Phi_E=\dfrac{Q_{enc}}{\epsilon_0}\Rightarrow 2EA'=\dfrac{\sigma A'}{\epsilon_0}$

By Gauss's law:

(e) Solve the result in (d) for the electric field.
$$E=\frac{\sigma}{2\epsilon_0}$$

(f) Apply your argument for the direction of the electric field in (a) to provide the vector electric field.

In the +x region of space, the electric field is given by: $\vec{E}_{x>0}=\dfrac{\sigma}{2\epsilon_0}\hat{x}$

In the -x region of space, the electric field is given by: $\vec{E}_{x<0}=-\dfrac{\sigma}{2\epsilon_0}\hat{x}$.

(5) Two infinite planes are separated by a distance d. Plane 1 has a surface charge $+\sigma$ while plane 2 has a surface charge $-\sigma$. Use Gauss's law to find $\vec{E}$ everywhere.

The geometry here is the same as in problem 4 except that one face of the cylinder is between the plates while the other is outside the plates. The face outside the plates has no flux through it. I can show this using the results of problem 4.

There are 3 regions to consider: (1) in the region x<0, (2) in the region 0<x<d and (3) in the region x>d.

In region 1, we have two electric fields which superimpose:
from the + plate at zero: $\vec{E}_+ = -\frac{\sigma}{2\epsilon_0}\hat{x}$ and from the - plate at d: $\vec{E}_- = \frac{\sigma}{2\epsilon_0}\hat{x}$. To find the electric field in region 1, add these two solutions:
$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{-\sigma}{2\epsilon_0}\hat{x} + \frac{\sigma}{2\epsilon_0}\hat{x} = 0\,\hat{x}$$
The electric flux through any face in the region x<0 is thus necessarily zero. Likewise, in region (3) the electric field is also zero. Thus the only flux will be between the two plates. The electric field in region 2 is then given by:

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\sigma}{2\epsilon_0}\hat{x} + \frac{\sigma}{2\epsilon_0}\hat{x} = \frac{\sigma}{\epsilon_0}\hat{x}$$

Thus the electric field will be zero outside the plates and twice as large between the plates. The presence of an oppositely charged plate means that there is no electric field outside the region between the two plates. This problem is extremely important to understand since this is the electric field between the plates of a parallel plate capacitor. It is somewhat important to note that while people may often say a capacitor has been charged up, you can see that actually the net charge across the plates of a capacitor is zero but there is a separation of charge which is the result of work having been done.