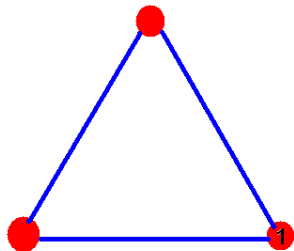Concepts: Electric Field, lines of force, charge density, dipole moment, electric dipole

(1) An equilateral triangle with each side of length 0.10 m has identical charges of $+q=1.0\mu C$. What is the net electrostatic **vector** force on charge 1?

(2) A point charge $q_1=-3.00\mu C$ is located at x=0. A second charge $q_2=+6.00\mu C$ is located at x=1.00 m. Find a point other than infinity where the electric field is zero.

(3) The electric dipole consists of a positive and a negative charge separated by a distance of 2a. Suppose in this case, your dipole had +q at x=a and -q at x=-a. Find an expression for the electric field along the y-axis. You should then be able to show that the electric field behaves as $E_x \approx -\dfrac{2kqa}{y^3}$ at distant points along the y-axis.

(4) Suppose in this case, your dipole had +q at x=a and -q at x=-a. Find an expression for the electric field along *the x-axis* at x>a. You should then be able to show that the electric field behaves as $E_x \approx \dfrac{4kqa}{x^3}$ at distant points along the *x* -axis. Then write the result in terms of the dipole moment.

(5) Suppose that you have a ring of radius r=a and total charge Q located in the x-y plane. What is the electric field for points along the symmetry axis of this ring? How does this field behave along the axis at distant points *along the symmetry axis*?

We have previously defined the electric force between two charges and we have talked about the law of charges. It turns out that it is less important to talk about the electric force than another quantity called the electric field. The **electric field** is a real physical entity and carries away energy from an accelerating electric charge.

The electric field is defined by:
$$\vec{E} \equiv \frac{\vec{F}_{electric}}{q}$$

Here, by definition, q is a **positive** test charge.

E points in the direction that a **positive test charge** would accelerate under the influence of an electric force. These "lines of force" can be sketched with a few rules:
(1) They point away from positive charges.
(2) They point towards negative charges.
(3) They don't intersect.
(4) They point normal to the surface of a conductor.
(5) The density of these lines is an indicator of the electric field strength
(6) A positive charge placed on one of the field lines accelerates in the line's direction.

I'll later show you how to draw these lines of force.
In more elegant terms, then, using the definition of force that we had in the last lecture, we can write the electric field as:

$$\vec{E}_p = \sum_{i=1}^{i=n} k \frac{q_i}{\left|\vec{r}_p - \vec{r}_i\right|^2} \hat{r}_{ip} = \sum_{i=1}^{i=n} k \frac{q_i}{\left|\vec{r}_{ip}\right|^2} \hat{r}_{ip}$$

Let's look at each of the symbols:
"n"=# of discrete charges in the system
q_i is the ith charge in the system.
"k" is coulomb's constant

$\vec{r}_p$ is the vector from the origin pointed towards the point p in space. This would also be the location of the positive test charge so the notation that we have developed is really the same here: the test charge is now charge p. $\vec{r}_i$ is the vector from the origin pointed towards the charge q_i in space. $\hat{r}_{ip}$ is the unit vector directed from the charge q_i towards the point p in space. Don't get hungup on the fact that a particular charge might not be located at the origin: apply the rules I've shown you in worksheet 1 and you will correctly calculate the electric field.

One thing that you want to know is just how do you calculate $\hat{r}_{ip}$. Here is the way although you have already seen this in worksheet 1. Firstly, $\vec{r}_{ip} \equiv \vec{r}_p - \vec{r}_i$. We can now, from this find the unit vector pretty easily. Again, in words: $\vec{r}_{ip}$ is the vector pointing from charge i toward point p in space. The unit vector pointing in this direction is given by:

$$\hat{r}_{ip} = \frac{\vec{r}_{ip}}{|\vec{r}_p - \vec{r}_i|}$$

As a side note: what are the dimensions (units) associated with a unit vector?

So, let me show you an example calculation.
Suppose $\vec{r}_p = x_p \hat{x} + y_p \hat{y} + z_p \hat{z}$ and $\vec{r}_i = x_i \hat{x} + y_i \hat{y} + z_i \hat{z}$.

Then,

$$\hat{r}_{ip} = \frac{\left(x_p - x_i\right)\hat{x} + \left(y_i - y_p\right)\hat{y} + \left(z_p - z_i\right)\hat{z}}{\sqrt{\left(x_p - x_i\right)^2 + \left(y_p - y_i\right)^2 + \left(z_p - z_i\right)^2}}$$

Here are some numerical examples:

Suppose $\vec{r}_1 = 1\hat{x} + 2\hat{y} + 3\hat{z}$ and $\vec{r}_p = 3\hat{x} + 2\hat{y} + 1\hat{z}$.
Then: $\vec{r}_{ip} = \vec{r}_p - \vec{r}_i = \left(3-1\right)\hat{x} + \left(2-2\right)\hat{y} + \left(1-3\right)\hat{z} = 2\hat{x} + 0\hat{y} - 2\hat{z}$
The unit vector is then: $\hat{r}_{ip} = \frac{2\hat{x} - 2\hat{z}}{\sqrt{2^2 + 2^2}} = \frac{2}{\sqrt{8}}\left(\hat{x} - \hat{z}\right) = \frac{1}{\sqrt{2}}\left(\hat{x} - \hat{z}\right)$

Another really easy example: Suppose $\vec{r}_i = 1\hat{x} + 0\hat{y} + 0\hat{z}$ and $\vec{r}_p = 0\hat{x} + 0\hat{y} + 0\hat{z}$.

$$\vec{r}_{ip} = \left(0-1\right)\hat{x} = -\hat{x} \text{ and } \hat{r}_{ip} = \frac{-\hat{x}}{1} = -\hat{x}$$

Note that $|\vec{r}_p - \vec{r}_i|$ is simply the distance between the charge and the point p. It is not hard from this to see that for special cases where the point of interest (p) is located at the origin. we have: $\vec{r}_{ip} = \vec{0} - \vec{r}_i = -\vec{r}_i$. One detail about notation: I'll write: $\vec{r}_{ip} \equiv \vec{r}_p - \vec{r}_i$ occasionally and you'll probably do the same. While this is more technical, in principle you might just find it an easier approach than having to resolve electric field components each time. This applies to charges that are discrete.

One final important point. I'll introduce in problems 3 and 4 the electric dipole moment. For a collection of j charges, we define the dipole moment as:

$$\vec{p} = \sum_{j=1}^{n} q_j \vec{r}_j$$

It's **important not to confuse** this $\vec{p}$ with the "p" which I'm using to designate the point in space. There is also one additional term which is going to be introduced later, namely the polarization of a material which is designated by $\vec{P}$ and is defined as the electric dipole moment per unit volume.

There are some details to the dipole moment which are worth noting. So long as the overall charge distribution is neutral, the dipole moment is coordinate independent. The dipole moment is related to the second term of a multipole expansion of the electric field. As it is normally used, the dipole moment refers to the approximate behavior of an ideal dipole at large distances from the dipole. By ideal, I mean that the distance between the positive and negative charge is insignificant compared to the distance to the point of interest in space. This, in turn, means the multipole expansion is valid. People use the dipole moment because it is often the largest contribution to the electric field. However, there are other terms in the multipole expansion that may be significant.

(1) An equilateral triangle with each side of length 0.10 m has identical charges of +q=1.0μC. What is the net electrostatic **vector** force on charge 1?
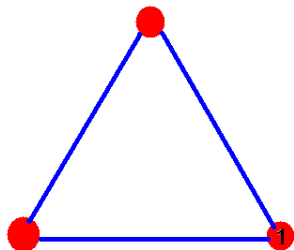
Solution: I want to show you 2 ways to do this problem. The first uses symmetry and is quicker. The second is the brute force method. You will find this problem on a spreadsheet. The force on any single charge is shown in red below:
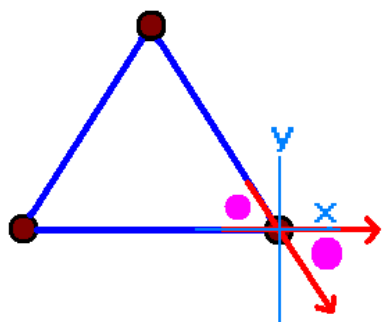
The little purple dots indicate the same angle which is 180/3=60 degrees. This is the angle θ that I'm using below. This only works because each of the charges is the same and each distance is the same.

I am indicating forces with the red arrows.

Thus, the "off-axis" force is going to be given by:

$$\vec{F}=F_x\hat{x}+F_y\hat{y}=F\left(\cos(\theta)\hat{x}-\sin(\theta)\hat{y}\right)$$

The total force on the single charge is then given by:

$$\vec{F}_{net}=F\hat{x}+F_x\hat{x}+F_y\hat{y}=F\left[(1+\cos(\theta))\hat{x}-\sin(\theta)\hat{y}\right]$$

We find the magnitude of this force from Coulomb's law:

$$|\vec{F}|=\left|k\frac{q_1q_2}{r^2}\right|=8.99\times10^9\left(\frac{1\times10^{-12}}{0.01}\right)=0.899N$$

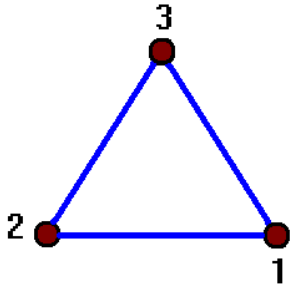We can now find the force on this charge: $\vec{F}=1.349\hat{x}-0.779\hat{y}N$.

The magnitude of this force is $|\vec{F}|=\sqrt{1.349^2+0.799^2}=1.558N$.

The angle which this force makes with respect to the x-axis is given by:

$$\tan(\phi)=\frac{F_y}{F_x}=\frac{-0.779}{1.349}\Rightarrow\phi=-30°$$.

The correct result here is -30 degrees.

Now let me show you the second (and, in my opinion the more powerful) way to do this.

I am putting labels on the charges as shown. I will let charge 1 be at the origin, so it has coordinates (0,0).

Probably the hardest part is to find the coordinates of charge #3. However you want to add it up, the x coordinate of the charge is $x=-0.05\,m$. The length of this vector pointing to charge 2 is 0.1m. Thus, the y coordinate is:

$$|\vec{r}_3|^2=x_3^2+y_3^2 \Rightarrow y_3^2=|\vec{r}_3|^2-x_3^2 \Rightarrow y_3=\sqrt{|\vec{r}_3|^2-x_3^2}$$
$$\Rightarrow y_3=\sqrt{.1^2-(-0.05)^2}=\sqrt{0.01-0.0025}=0.0867$$

The various vectors are then:

$$\vec{r}_1=0\hat{x}+0\hat{y};\vec{r}_3=-0.05\hat{x}+0.08667\hat{y};\vec{r}_2=-0.1\hat{x}+0\hat{y};\vec{r}_p=0\hat{x}+0\hat{y}$$

Now we're going to need to calculate this:

$$\vec{F}_p=\sum_{\substack{i=1\\i\neq p}}^{n} k\frac{q_iq_p}{|\vec{r}_{ip}|^2}\hat{r}_{ip}$$

That means if we're calculating the force on charge 1, we need the following:

$$\vec{F}_{(p=1)}=\sum_{\substack{i=1\\i\neq(p=1)}}^{n} k\frac{q_iq_{p=1}}{|\vec{r}_{ip}|^2}\hat{r}_{ip\ (p=1)}=k\frac{q_1q_2}{|\vec{r}_{21}|^2}\hat{r}_{21}+k\frac{q_3q_1}{|\vec{r}_{31}|^2}\hat{r}_{31}$$

We are going to need to calculate the various vectors involved here. I'm going to try to show this in detail here.

$$\vec{r}_{21}=\vec{r}_1-\vec{r}_2=[0\hat{x}+0\hat{y}]-[-0.1\hat{x}+0\hat{y}]=0.1\hat{x}+0\hat{y}:\hat{r}_{21}=\frac{\vec{r}_{21}}{|\vec{r}_{21}|}=\frac{\hat{x}}{|\hat{x}|}=\hat{x}$$

$$\vec{r}_{31}=\vec{r}_1-\vec{r}_3=[0\hat{x}+0\hat{y}]-[-0.05\hat{x}+0.0866\hat{y}]=0.05\hat{x}-0.0866\hat{y}$$
$$\hat{r}_{31}=\frac{\vec{r}_{31}}{|\vec{r}_{31}|}=\frac{0.05\hat{x}-0.08667\hat{y}}{\sqrt{(-0.05)^2+(0.08667)^2}}=\frac{0.05\hat{x}-0.08667\hat{y}}{.1}=0.5\hat{x}-0.8667\hat{y}=0.5\hat{x}-\frac{\sqrt{3}}{2}\hat{y}$$

The electric force at charge 1 ("p") due to the other two charges is then:

$$\vec{F}_1=k\sum_{\substack{i=2\\i\neq1}}^{3}\frac{q_iq_p}{|\vec{r}_{i1}|^2}\hat{r}_{i1}$$

$$\vec{F}_1=kq_1\left[\frac{q_2}{.01}\hat{x}+\frac{q_3}{.01}\left(0.5\hat{x}-\frac{\sqrt{3}}{2}\hat{y}\right)\right]=\frac{kq^2}{0.01}[1.5\hat{x}-0.866\hat{y}]$$

$$\Rightarrow\vec{F}_1=8.99x10^9x1x10^{-12}[150\hat{x}-86.6\hat{y}]=1349x10^{-3}\hat{x}-779x10^{-3}\hat{y}=1.35\hat{x}-0.78\hat{y}\,N$$

$$|\vec{F}_1|=\sqrt{(1.35)^2+(-0.78)^2}=1.56\,N$$

This force makes the same angle with respect to the +x axis as before:

$$\tan(\phi)=\frac{F_y}{F_x}=\frac{-0.78}{1.35}\Rightarrow\phi=-30°$$

We could have found the same result by calculating the electric field at charge 1 due to charges 2 and 3. The electric field at this point is given by:

$$\vec{F}_p = \sum_{\substack{i=1 \\ i \neq p}}^{n} k \frac{q_i q_p}{\left|\vec{r}_{ip}\right|^2} \hat{r}_{ip}$$

$$\vec{E}_1 = k\left[\frac{q_2}{.01}\hat{x} + \frac{q_3}{.01}\left(0.5\,\hat{x} - \frac{\sqrt{3}}{2}\,\hat{y}\right)\right] = \frac{kq}{0.01}\left[1.5\,\hat{x} - 0.866\,\hat{y}\right]$$

$$\Rightarrow \vec{E}_1 = 8.99\text{x}10^9 \text{x} 1\text{x}10^{-6}\left[150\,\hat{x} - 86.6\,\hat{y}\right] = \frac{8990}{0.01}\left[1.5\,\hat{x} - 0.866\,\hat{y}\right] = \left[1.35\text{x}10^3\,\hat{x} - 0.78\text{x}19^6\,\hat{y}\right]\frac{N}{C}$$

or

$$\vec{E}_p = 1.35\text{x}10^6\,\hat{x} - 0.78\text{x}10^6\,\hat{y}\,\frac{N}{C}$$

Then to get the force, multiply the electric field by the charge at point 1.

Notice that electric fields can get pretty large. But, a Newton of force from electrostatic charges is also pretty large.

(2) A point charge $q_1=-3.00\mu C$ is located at $x=0$. A second charge $q_2=+6.00\mu C$ is located at $x=1.00$ m. Find a point other than infinity where the electric field is zero.

The electric field is defined by:

$$\vec{E}_p=\sum_{\substack{i=1 \\ i\neq p}}^{n} k\frac{q_i}{|\vec{r}_{ip}|^2}\hat{r}_{ip}$$

where p represents a point in space.

We locate the initial charge at $\vec{r}_1=0\hat{x}$ and the second charge at $\vec{r}_2=1\hat{x}$.
The vector pointing to p is given (in two dimensions) by:

$$\vec{r}_p=x_p\hat{x}+y_p\hat{y}$$

The electric field at any point in 2-D space is then given by:

$$\vec{E}_p=k\frac{q_1}{|\vec{r}_{1p}|^2}\hat{r}_{1p}+k\frac{q_2}{|\vec{r}_{12}|^2}\hat{r}_{2p}$$

We can easily form each of these vectors now.

$$\vec{r}_{1p}=\vec{r}_p-\vec{r}_1=x_p\hat{x}+y_p\hat{y}-0\hat{x}-0\hat{y}=x_p\hat{x}+y_p\hat{y}:\hat{r}_{1p}=\frac{x_p\hat{x}+y_p\hat{y}}{\sqrt{x_p^2+y_p^2}}$$

$$\vec{r}_{2p}=\vec{r}_p-\vec{r}_2=x_p\hat{x}+y_p\hat{y}-1\hat{x}-0\hat{y}=(x_p-1)\hat{x}+y_p\hat{y}:\hat{r}_{2p}=\frac{(x_p-1)\hat{x}+y_p\hat{y}}{\sqrt{(x_p-1)^2+y_p^2}}$$

We now form the electric field:

$$\vec{E}_p=kq_1\frac{1}{(x_p^2+y_p^2)}\frac{x_p\hat{x}+y_p\hat{y}}{\sqrt{x_p^2+y_p^2}}+kq_2\frac{1}{((x_p-1)^2+y_p^2)}\frac{(x_p-1)\hat{x}+y_p\hat{y}}{\sqrt{(x_p-1)^2+y_p^2}}$$

We need to solve this for $\vec{E}_p=\vec{0}$

In this particular problem, we notice $q_2=-2q_1$. Thus:

$$\vec{0}=q_1\frac{1}{(x_p^2+y_p^2)}\frac{x_p\hat{x}+y_p\hat{y}}{\sqrt{x_p^2+y_p^2}}-2q_1\frac{1}{((x_p-1)^2+y_p^2)}\frac{(x_p-1)\hat{x}+y_p\hat{y}}{\sqrt{(x_p-1)^2+y_p^2}}$$

$$\vec{0}=\frac{1}{(x_p^2+y_p^2)}\frac{x_p\hat{x}+y_p\hat{y}}{\sqrt{x_p^2+y_p^2}}-2\frac{1}{((x_p-1)^2+y_p^2)}\frac{(x_p-1)\hat{x}+y_p\hat{y}}{\sqrt{(x_p-1)^2+y_p^2}}$$

$$\frac{1}{(x_p^2+y_p^2)}\frac{x_p\hat{x}+y_p\hat{y}}{\sqrt{x_p^2+y_p^2}}=2\frac{1}{((x_p-1)^2+y_p^2)}\frac{(x_p-1)\hat{x}+y_p\hat{y}}{\sqrt{(x_p-1)^2+y_p^2}}$$

$$\hat{x}:\frac{x_p}{[x_p^2+y_p^2]^{3/2}}=2\frac{(x_p-1)}{[(x_p-1)^2+y_p^2]^{3/2}}$$

$$\hat{y}:\frac{y_p}{[x_p^2+y_p^2]^{3/2}}=2\frac{y_p}{[(x_p-1)^2+y_p^2]^{3/2}}$$

It is easy to see that the y equation is satisfied with $y_p=0$. This could have also come from symmetry (in past versions of this problem I merely assumed this).

Use this in the x-equation:

$$\frac{x_p}{x_p^3}=2\frac{\left(x_p-1\right)}{\left[\left(x_p-1\right)^2\right]^{3/2}}=2\frac{x_p-1}{\left[x_p-1\right]^3}\Rightarrow 2x_p^2=(x_p-1)^2\Rightarrow\sqrt{2}x_p=\pm\left(x_p-1\right)$$

$$+:\sqrt{2}x_p=x_p-1\Rightarrow x_p(1-\sqrt{2})-1=0\Rightarrow x_p=\frac{1}{\left(1-\sqrt{2}\right)}$$

$$-:\sqrt{2}x_p=-x_p+1\Rightarrow x_p\left(1+\sqrt{2}\right)-1=0\Rightarrow x_p=\frac{1}{\left(1+\sqrt{2}\right)}$$

The two solutions are thus obtained. Let me confirm that E is zero for each of these points. If it's not we'll need to discard one solution.

On to the actual answer for the solutions:

if (solution 1) $\quad x_p=\dfrac{1}{1+\sqrt{2}}=0.4142$, the electric field is not zero.

if: (solution 2) $\quad x_p=\dfrac{1}{1-\sqrt{2}}=-2.4142$, the electric field is zero.

You can check these results using the spreadsheet on our website by choosing a test charge of +1 located at each of the two solutions for x. I have provided the spreadsheet showing the "zero" solution for you.
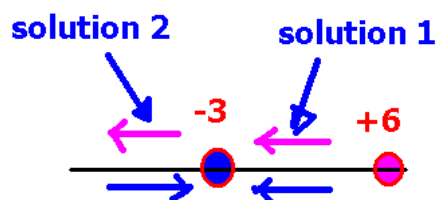
What this sketch above shows is that between the two charges, it is impossible to have zero electric field (solution 1 is between the two charges). Solution 2 however is in a region where it is possible to have zero electric field. The only other region where you might consider is outside (at positive x). However, you will never get zero there because the +6 charge is not only larger than the -3 charge, but it is also always closer to the point in that region.

Now, how else could this be solved? This is a quicker (and not nearly so mathematically clean here): You could say this: the electric field at a point along the x-axis is:

$$\vec{E}_p=k\frac{q_1}{x_{1p}^2}\hat{r}_{1p}+k\frac{q_2}{x_{2p}^2}\hat{r}_{2p}$$

Now let's lose the vector notation. This will increase the number of incorrect solutions but that's the price you'll need to pay for this. Thus, we have:

$$\frac{q_1}{x_{1p}^2}-\frac{q_2}{x_{2p}^2}=0 \quad\text{or}\quad \frac{q_1}{x_{1p}^2}+\frac{q_2}{x_{2p}^2}=0$$

here, $q_2=-2q_1$ so this gives:

$$\frac{1}{x_{1p}^2}+\frac{2}{x_{2p}^2}=0 \quad\text{or}\quad \frac{1}{x_{1p}^2}-\frac{2}{x_{2p}^2}=0$$

We don't need to mess with the first result. It won't give real x values. Solving the second equation then gives:

$$\frac{1}{x_{1p}^2}=\frac{2}{x_{2p}^2}\Rightarrow\frac{1}{x^2}=\frac{2}{\left(x-1\right)^2}\Rightarrow 2x^2=\left(x-1\right)^2$$

The rest of the solution proceeds as before.

$$2x_p^2=(x_p-1)^2 \Rightarrow \sqrt{2}\,x_p=\pm(x_p-1)$$

$$+ : \sqrt{2}\,x_p=x_p-1 \Rightarrow x_p(1-\sqrt{2})-1=0 \Rightarrow x_p=\frac{1}{(1-\sqrt{2})}$$

$$- : \sqrt{2}\,x_p=-x_p+1 \Rightarrow x_p(1+\sqrt{2})-1=0 \Rightarrow x_p=\frac{1}{(1+\sqrt{2})}$$

The following problem is quite important. Be sure you understand it.

(3) The electric dipole consists of a positive and a negative charge separated by a distance of 2a. Suppose in this case, your dipole had +q at x=a and -q at x=-a. Find an expression for the electric field along the y-axis. You should then be able to show that the electric field behaves as $E_x \approx -\dfrac{2kqa}{y^3}$ at distant points along the y-axis.

We begin with the definition of the electric field:
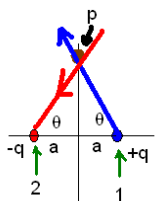
$$\vec{E}_p=\sum_{i=1}^{i=n} k \frac{q_i}{|\vec{r}_i-\vec{r}_p|^2}\hat{r}_{ip}$$

Now we need to obtain the various vectors involved.

$$\vec{r}_1=a\hat{x} : \vec{r}_2=-a\hat{x} : \vec{r}_p=y_p\hat{y}$$

$$\vec{r}_{1p}=\vec{r}_p-\vec{r}_1=-a\hat{x}+y_p\hat{h} : \vec{r}_{2p}=\vec{r}_p-\vec{r}_2=a\hat{x}+y_p\hat{y}$$

$$\hat{r}_{1p}=\frac{\vec{r}_{1p}}{|\vec{r}_{1p}|}=\frac{-a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}} : \hat{r}_{2p}=\frac{\vec{r}_{2p}}{|\vec{r}_{2p}|}=\frac{a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}}$$

Now we need to use these in the definition of the electric field.

$$\vec{E}_p=k\frac{q_1}{|\vec{r}_{1p}|^2}\hat{r}_{1p}+k\frac{q_2}{|\vec{r}_{2p}|^2}\hat{r}_{2p} \quad \text{so here:} \quad \vec{E}_p=k\frac{q_1}{a^2+y_p^2}\frac{-a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}}+k\frac{q_2}{a^2+y_p^2}\frac{a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}}$$

We thus have: (letting q_1 be the same magnitude as q_2 but of opposite sign):

$$\vec{E}_p=kq\left[\frac{1}{a^2+y_p^2}\frac{-a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}}-\frac{1}{a^2+y_p^2}\frac{a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}}\right]$$

$$\vec{E}_p=kq\left[\frac{1}{a^2+y_p^2}\frac{-2a\hat{x}}{\sqrt{a^2+y_p^2}}\right]=-\frac{2kqa}{(a^2+y_p^2)^{3/2}}\hat{x}$$

This is the actual answer. Now let's look at how this behaves for y>>a. The expansion is:

$$(1\pm x)^n=1\pm nx+\frac{n(n-1)}{2!}x^2\pm\frac{n(n-1)(n-2)}{3!}x^2+...;x^2<1$$

Also note: $(1\pm x)^{-n}=1\mp nx+\frac{n(n-1)}{2!}x^2\mp\frac{n(n-1)(n-2)}{3!}x^2+...;x^2<1$

What you want to do is to divide by what is big. Here, that would be y_p.

$$\vec{E}_p = \frac{-2kqa}{|y_p|^3}\left[1+\left(\frac{a}{y_p}\right)^2\right]^{-3/2}\hat{x} \approx \frac{-2kqa}{|y_p|^3}\left[1-\frac{3}{2}\left(\frac{a}{y_p}\right)^2 + ...\right]\hat{x} \approx \frac{-2kqa}{|y_p|^3}\hat{x}$$

The electric field at large distances along the perpendicular bisector of the dipole is:

$$\vec{E}_p \approx \frac{-2kqa}{|y_p|^3}\hat{x}$$

Both of these results are extremely important for systems involving electric dipoles! It is also indeed very interesting to see that the dipole falls off as $1/y^3$ at large distances. The term p=2qa is called the magnitude of the electric dipole moment.

With the approximate field above, we can define the dipole moment for 2 equal and opposite charges with charges and locations:

$$(+q;+a,0,0) \text{ and } (-q;-a,0,0)$$
$$\text{as:}$$

$$\vec{p} = \sum_{j=1}^{2} q_j\vec{r}_j = qa\,\hat{x}+(-q)(-a\hat{x}) = q(2a\hat{x}) = q\vec{d}$$

where $\vec{d}$ is the vector pointing ***from the negative charge towards the positive charge***.

With this, the electric field along the perpendicular bisector of the electric dipole becomes:

$$\vec{E}_p \approx -k\frac{\vec{p}}{|y_p|^3}$$

This is extremely important because it defines the electric dipole. Students are warned that many (about ½) of the undergraduate general chemistry text books get this wrong.

Now, what if we have the same diople but wanted to find the field at an arbitrary location in the x-y plane.

$$\vec{r}_1 = a\hat{x} : \vec{r}_2 = -a\hat{x} : \vec{r}_p = x_p\hat{x} + y_p\hat{y}$$

$$\vec{r}_{1p} = \vec{r}_p - \vec{r}_1 = (x_p - a)\hat{x} + y_p\hat{h} : \vec{r}_{2p} = \vec{r}_p - \vec{r}_2 = (x_p + a)\hat{x} + y_p\hat{y}$$

$$\hat{r}_{1p} = \frac{\vec{r}_{1p}}{|\vec{r}_{1p}|} = \frac{(x_p - a)\hat{x} + y_p\hat{y}}{\sqrt{(x_p - a)^2 + y_p^2}} : \hat{r}_{2p} = \frac{\vec{r}_{2p}}{|\vec{r}_{2p}|} = \frac{(x_p + a)\hat{x} + y_p\hat{y}}{\sqrt{(x_p + a)^2 + y_p^2}}$$

$$\vec{E}_p = kq\left[\frac{1}{(x_p - a)^2 + y_p^2}\frac{(x_p - a)\hat{x} + y_p\hat{y}}{\sqrt{(x_p - a)^2 + y_p^2}} - \frac{1}{(x_p - a)^2 + y_p^2}\frac{(x_p + a)\hat{x} + y_p\hat{y}}{\sqrt{(x_p + a)^2 + y_p^2}}\right]$$

$$\vec{E}_p = kq\left[\frac{(x_p - a)\hat{x} + y_p\hat{y}}{\left[(x_p - a)^2 + y_p^2\right]^{3/2}} - \frac{(x_p + a)\hat{x} + y_p\hat{y}}{\left[(x_p + a)^2 + y_p^2\right]^{3/2}}\right]$$

$$\vec{E}_p = kq\left[\frac{(x_p-a)\hat{x}+y_p\hat{y}}{\left[(x_p-a)^2+y_p^2\right]^{3/2}} - \frac{(x_p+a)\hat{x}+y_p\hat{y}}{\left[(x_p+a)^2+y_p^2\right]^{3/2}}\right]$$

To get the behavior at large distances,

Note that if your dipole is not located at the origin, then do a coordinate translation (and maybe a rotation) to put it there. However, with the more strict formulation, this is not essential so let's see now what the electric field is for two equal but opposite charges located at two random points in the x-y plane.

$$1:(+q,x_1,x_2):2:(-q,x_2,y_2):p:(x_p,y_p)$$
$$\vec{r}_p = x_p\hat{x}+y_p\hat{y}:\vec{r}_1=x_1\hat{x}+y_1\hat{y}:\vec{r}_2=x_2\hat{x}+y_2\hat{y}$$
$$\vec{r}_{1p}=(x_p-x_1)\hat{x}+(y_p-y_1)\hat{y}:\vec{r}_{2p}=(x_p-x_2)\hat{x}+(y_p-y_2)\hat{y}$$

$$\vec{E}_p = kq\left[\frac{(x_p-x_1)\hat{x}+(y_p-y_1)\hat{y}}{\left[(x_p-x_1)^2+(y_p-y_1)^2\right]^{3/2}} - \frac{(x_p-x_2)\hat{x}+(y_p-y_2)\hat{y}}{\left[(x_p-x_2)^2+(y_p-y_2)^2\right]^{3/2}}\right]$$

At this point, step by step, you become more restrictive but the result above is a general result.
So let both charges lie along the x-axis. With this restriction, we have:

$$\vec{E}_p = kq\left[\frac{(x_p-x_1)\hat{x}+(y_p)\hat{y}}{\left[(x_p-x_1)^2+(y_p)^2\right]^{3/2}} - \frac{(x_p-x_2)\hat{x}+(y_p)\hat{y}}{\left[(x_p-x_2)^2+(y_p)^2\right]^{3/2}}\right]$$

$$\vec{E}_p = kq\left[\frac{(x_p-a)\hat{x}+(y_p)\hat{y}}{\left[(x_p-a)^2+(y_p)^2\right]^{3/2}} - \frac{(x_p+a)\hat{x}+(y_p)\hat{y}}{\left[(x_p+a)^2+(y_p)^2\right]^{3/2}}\right]$$

$$\vec{E}_p = kq\left[\frac{\vec{r}_p-\vec{a}}{\left[\vec{r}_p^2-2\vec{r}_p\cdot\vec{a}+a^2\right]^{3/2}} - \frac{\vec{r}_p+\vec{a}}{\left[\vec{r}_p^2+2\vec{r}_p\cdot\vec{a}+a^2\right]^{3/2}}\right] \quad ; \quad \vec{a}=a\,\hat{x}$$

$$\text{let} \quad |\vec{r}_p|\gg|\vec{a}|$$

$$\vec{E}_p = \frac{kq}{|\vec{r}_p|^{3/2}}\left[\frac{\vec{r}_p-\vec{a}}{\left[1-2\frac{\hat{r}_p\cdot\vec{a}}{r_p}+\left(\frac{a}{r_p}\right)^2\right]^{3/2}} - \frac{\vec{r}_p+\vec{a}}{\left[1+2\frac{\hat{r}_p\cdot\vec{a}}{r_p}+\left(\frac{a}{r_p}\right)^2\right]^{3/2}}\right]$$

$$\vec{E}_p = \frac{kq}{|\vec{r}_p|^{3/2}}\left[\frac{\vec{r}_p-\vec{a}}{\left[1-2\frac{a}{r_p}\cos\theta_p+\left(\frac{a}{r_p}\right)^2\right]^{3/2}} - \frac{\vec{r}_p+\vec{a}}{\left[1+2\frac{a}{r_p}\cos\theta_p+\left(\frac{a}{r_p}\right)^2\right]^{3/2}}\right]$$

$$(1\pm x)^{-n}=1\mp nx+\frac{n(n-1)}{2!}x^2\mp\frac{n(n-1)(n-2)}{3!}x^2+\ldots;x^2<1$$

$$\vec{E}_p = \frac{kq}{|\vec{r}_p|^{3/2}}\left[(\vec{r}_p - \vec{a})\left[1 - \frac{3}{2}\left(-2\frac{a}{r_p}\cos\theta_p + \left(\frac{a}{r_p}\right)^2\right)\right] - (\vec{r}_p + \vec{a})\left[1 - \frac{3}{2}\left(2\frac{a}{r_p}\cos\theta_p + \left(\frac{a}{r_p}\right)^2\right)\right]\right]$$

To lowest order:

$$\vec{E}_p \approx \frac{k}{|\vec{r}_p|^{3/2}}\left[-2q\vec{a} + q\vec{r}_p\left(\frac{6a}{r_p}\cos\theta - 3\left(\frac{a}{r_p}\right)^2\right)\right] = \frac{k}{|\vec{r}_p|^{3/2}}\left[-\vec{p} + q\vec{r}_p\left(\frac{6a}{r_p}\cos\theta\right)\right] = \frac{k}{|\vec{r}_p|^{3/2}}\left[(3\vec{p}\cdot\hat{r}_p)\hat{r}_p - \vec{p}\right]$$

So

$$\vec{E}_p \approx \frac{k}{|\vec{r}_p|^{3/2}}\left[(3\vec{p}\cdot\hat{r}_p)\hat{r}_p - \vec{p}\right]$$

Essentially this approximation of the dipole has reduced the dipole down to the extent that the distance between charges is insignificant compared to the point in space. It may not be valid for distributions where the net charge is not zero and also in the case that you are close to the dipole, relative to the distance between the charges. The important thing to keep in mind here is this: the dipole moment is only one of the terms in the multipole expansion and usually it is the predominate term. Continuous charge distributions are most easily done with calculus except is some special circumstances. So you are left with this:

$$\vec{p} = \sum_{\text{all } q_i} q_i \vec{r}_i$$

which is certainly an untidy way to write this but it contains the explicit understanding that $\vec{r}_i$ points towards spatial regions containing charges that are part of the charge distribution, but keep in mind that discrete charge distributions and continuous charge distributions are different entities entirely.

(4) Suppose in this case, your dipole had +q at x=a and -q at x=-a. Find an expression for the electric field along *the x-axis* at x>a. You should then be able to show that the electric field behaves as $\vec{E}_p \approx \dfrac{4kqa\hat{x}}{x_p^3} = \dfrac{2k\vec{p}}{x_p^3}$ at distant points along the *x*-axis.

In the previous problem, we had:

$$\vec{E}_p = kq\left[\frac{(x_p-a)\hat{x}+(y_p)\hat{y}}{\left[(x_p-a)^2+(y_p)^2\right]^{3/2}} - \frac{(x_p+a)\hat{x}+(y_p)\hat{y}}{\left[(x_p+a)^2+(y_p)^2\right]^{3/2}}\right]$$

Here, allow y_p to be zero. then this simplifies to become:

$$\vec{E}_p = kq\left[\frac{(x_p-a)\hat{x}}{\left[(x_p-a)^2\right]^{3/2}} - \frac{(x_p+a)\hat{x}}{\left[(x_p+a)^2\right]^{3/2}}\right]$$

Consider the case where $x_p-a>0$:

$$\vec{E}_p = kq\left[\frac{1}{(x_p-a)^2} - \frac{1}{(x_p+a)^2}\right]\hat{x} = \frac{kq}{(x_p-a)^2(x_p+a)^2}\left[x_p+2x_pa+a^2-x_p+2x_pa-a^2\right]\hat{x}$$

$$= \frac{kq}{(x_p-a)^2(x_p+a)^2}\left[4x_pa\right]\hat{x} \approx \frac{4kqa\hat{x}}{x_p^3} = \frac{2k\vec{p}}{x_p^3}$$

You could also use the result above: $\vec{E}_p \approx \dfrac{k}{|\vec{r}_p|^{3/2}}\left[(3\vec{p}\cdot\hat{r}_p)\hat{r}_p - \vec{p}\right]$ .In this case: $\vec{E}_p \approx \dfrac{k}{x_p^3}\left[2\vec{p}\right]$.

Remember, however, our expression for the dipole, $\vec{E}_p \approx \dfrac{k}{|\vec{r}_p|^{3/2}}\left[(3\vec{p}\cdot\hat{r}_p)\hat{r}_p - \vec{p}\right]$, is really only valid for r>>a whereas doing the exact calculation is always valid (since it is without approximation). This means that you can not always start with the field for the dipole to represent any dipole you run into! However at those times when you are in the correct region for approximation, it is appropriate to use this result.

(5) Suppose that you have a ring of radius r=a and total charge Q located in the x-y plane. What is the electric field for points along the symmetry axis of this ring? How does this field behave along the axis at distant points *along the symmetry axis*?

Non-calculus version
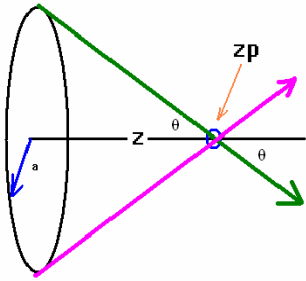
This picture showing this particular situation is to the left. The symmetry of the problem allows me to say that the only components of the electric field which survive will lie along the z-axis (i.e. the off-axis components of the electric field cancel) at points along the z-axis.

In this case, then, we have $\vec{E}_{total} = \sum_j \vec{E}_j = \sum_j |\vec{E}_j| \hat{r}_{jp}$.

The angle Θ is the same no matter where on the ring you look from the symmetry axis (at a fixed z_p) . Also, the distance from the ring to the point z_p is the same for every point along the ring. To determine the electric field, write the charge on the ring in terms of the *charge density* on the ring. If you consider that the ring has a total length given by: $2\pi a$, the total charge Q on the ring is given by: $Q = (2\pi a)\lambda$ where I am representing a linear charge density here by λ. The electric field from a very small section at the top of the ring is

given by: $\vec{E}_{j_+} = \dfrac{kq_j}{(a^2 + z_p^2)} \dfrac{z_p\hat{z} - a\hat{y}}{\sqrt{z_p^2 + a^2}}$ where the subscript "+" means I've picked the point from

the top of the ring. The electric field coming from a point on the bottom of the ring (exactly opposite from the previous position) is given by:

$$\vec{E}_{j_-} = \dfrac{kq_j}{(a^2 + z_p^2)} \dfrac{z_p\hat{z} + a\hat{y}}{\sqrt{z_p^2 + a^2}}$$

If I add these two electric fields, I get the result:

$$\vec{E}_{j_+} + \vec{E}_{j_-} = \dfrac{2kq_j}{[a^2 + z_p^2]^{3/2}} z_p\hat{z}$$

Now you need to determine how many such charge pairs there are on the ring. If you let the small charge q_j be represented by $q_j = \lambda(a(\Delta\phi))$ where $\Delta\phi$ represents a small angle, then we can rewrite the electric field as:

$$\vec{E}_{j_+} + \vec{E}_{j_-} = \dfrac{2k\lambda a(\Delta\phi)}{[a^2 + z_p^2]^{3/2}} z_p\hat{z}$$

If we now let $\Delta\phi$ represent ½ of the total angle of the ring (which is 2π and remember, I'm adding up charge pairs here), the electric field from the entire ring becomes:

$$\vec{E}_{j_+} + \vec{E}_{j_-} = \dfrac{2k\lambda a(\pi)}{[a^2 + z_p^2]^{3/2}} z_p\hat{z}$$

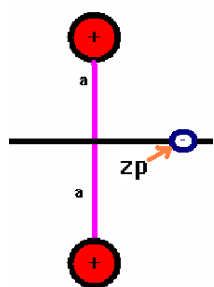In terms of the total charge Q placed upon the ring, we thus have:

$$\vec{E}_{j_+} + \vec{E}_{j_-} = \dfrac{kQz_p}{[a^2 + z_p^2]^{3/2}} \hat{z}$$

Let's also look at how this behaves as x gets large.

$$(a^2 + z_p^2)^{-3/2} = z_p^3\left(1 + \dfrac{a^2}{z_p^2}\right)^{3/2} = z_p^3\left(1 + \dfrac{3}{2}\dfrac{a^2}{z_p^2} + ...\right) \approx z_p^3$$

At large distances, $\vec{E}_p \approx \dfrac{kQ}{z_p^2}\hat{z}$ (the ring looks a lot like a point charge).

Here is a nice application of what you have learned that also ties some things together!

Suppose you have a crystal which has two positive charges located as shown and an electron is located along the symmetry axis between the two charges at a distance z_p from the center which is very small compared to a. Let's see what happens.

This problem is unlike the dipole problem in that each of the charges is the same. However, looking at the non-calculus approach to the ring problem (problem 5), it is immediately apparent what the electric field is along the symmetry axis. The electric field is given by:

$$\vec{E}_{j_+ + j_-} = \frac{k(2q_j)}{\left[a^2 + z_p^2\right]} \frac{z_p}{\sqrt{z_p^2 + a^2}} \hat{z} = \frac{2kq\,z_p}{\left[a^2 + z_p^2\right]^{3/2}} \hat{z}$$

Now we're going to look at this expression in the limit that $|z_p| \ll a$. We again use the binomial expansion but we need to rewrite the denominator slightly.

$$\left[a^2 + z_p^2\right]^{3/2} = a^3 \left[1 + \left(\frac{z_p}{a}\right)^2\right]^{3/2} \approx a^3 \left(1 + \frac{3}{2}\left(\frac{z_p}{a}\right)^2 + \ldots\right)$$

The leading term is then a^3 which gives us the approximate electric field at the center as:

$$\vec{E}_p \approx \frac{2keq}{a^3} z_p \hat{z}$$

Now let's find the electrostatic force on the electron which is trapped in such a situation. This is easily seen to be given by:

$$\vec{F} = \left(q_{electron}\right)\vec{E}_p = -e\vec{E}_p = \frac{-2keq}{a^3} z_p \hat{z}$$

This force is linear in the displacement variable and restoring. If you compare this force to the Hooke's law force $\left(\vec{F} = -Kx\hat{x}\right)$ then you would expect to see the electron oscillate with simple harmonic oscillation and thus would have a frequency given by:

$$\omega = \sqrt{\frac{K}{m_e}} = \sqrt{\frac{2keq}{a^3 m_3}} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{2keq}{a^3 m_e}}$$

You often hear that molecules act like springs connected to masses but this really shows the effect. The electron will oscillate (and thus, it will store energy). The problem is that this is a classical calculation. It is, however, very easy at this point, with a little bit of quantum mechanics to obtain an energy spectrum for the electron trapped between two positively charged ions like this!

The energy spectrum will be given by:

$$E_n = \hbar \omega \left(n + \frac{1}{2} \right); n = 1,2,3,\dots; \omega = \sqrt{\frac{K}{m_e}}$$

Here, you also see Planck's constant which is given by: $\hbar = \frac{h}{2\pi} = 1.0546 \times 10^{-34} \, J \cdot s$
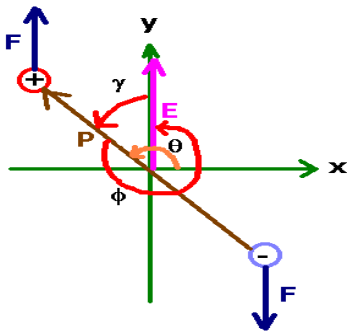
Here is another nice application related to the electric dipole. Suppose that we apply a uniform electric field along the y-axis of the dipole in problem 4. Assume an external electric field is given by:

$$\vec{E}=E\hat{y}$$

The angle between $\vec{p}$ and $\vec{E}$ is φ. The angle between $\vec{E}$ and $\vec{p}$ is γ. These two angles are related by: $\phi+\gamma=360^0$. The angle between the positive x-axis and $\vec{p}$ is θ.
The coordinates of the charges are:

$$\vec{r}_+=a\cos(\theta)\hat{x}+a\sin(\theta)\hat{y}:\vec{r}_-=a\cos(\theta+180^0)\hat{x}+a\sin(\theta+180^0)\hat{y}$$

The torque on the positive charge is given by:

$$\vec{\tau}_+=\vec{r}_+X\vec{F}_+=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\a\cos(\theta)&a\sin(\theta)&0\\0&E|q|&0\end{vmatrix}=\hat{x}(0)-\hat{y}(0)+\hat{z}\big(E|q|a\cos(\theta)\big)=E|q|a\cos(\theta)\hat{z}$$

Since $\cos(\beta)=-\cos(\beta+180^0)$, and the fact that the negative charge has a negative sign, the torque from the negative charge is the same as for the positive charge.

$$\vec{\tau}_-=\vec{\tau}_+\Rightarrow\vec{\Gamma}=-2Ea|q|\cos(\theta)\hat{z}=|\vec{p}||\vec{E}|\cos(\theta)\hat{z}$$

Now, there is also another connection:

$$\gamma+90^0=\theta\Rightarrow\gamma=\theta-90^0$$
$$360^0-\gamma=\phi\Rightarrow360^0-\theta+90^0=450^0-\theta=\phi\Rightarrow\theta=450^0-\phi$$

We thus have the net torque on the dipole given as:

$$\Gamma=|\vec{p}||\vec{E}|\cos(450^0-\phi)\hat{z}=|\vec{p}||\vec{E}|\big[\cos(450^0)\cos(\phi)+\sin(450^0)\sin(\phi)\big]\hat{z}$$
$$\Rightarrow\vec{\Gamma}=|p||E|\sin(\phi)\hat{z}=\vec{p}X\vec{E}$$

where the angle is measured starting with the positive $\vec{p}$ axis and rotating around in the positive manner (counterclockwise). On the other hand, if you want to relate this to the angle γ which starts along the Positive E direction and rotates counterclockwise towards p, then you have

$$\sin(\phi)=\sin(360^0-\gamma)=\sin(360^0)\cos(\gamma)\sin(\gamma)\cos(360^0)=-\sin(\gamma)$$

Thus the torque is given by:

$$\vec{\Gamma}=-|\vec{p}||\vec{E}|\sin(\gamma)\hat{z}=\vec{p}X\vec{E}$$

Now here is why I worry so much about the sign of this torque: if the sign is wrong, simple harmonic oscillation won't result from the analysis below. In particular, you want to fix yourself onto the electric field vector and watch the dipole oscillate about your reference frame, rather than fixing yourself on the dipole and watching the electric field oscillate. According to Newton's laws, we have that a torque produces an angular acceleration:

$$\Gamma=I\alpha$$ So the equation of motion is given by:

$$-|\vec{p}||\vec{E}|\sin(\gamma)=I\alpha\Rightarrow\alpha=\frac{|\vec{p}||\vec{E}|}{I}\sin(\gamma)=0$$

Now if you consider only small angles, then:
$$\sin(\gamma) \approx \gamma$$

This means that simple harmonic oscillation will result with a frequency of oscillation given by:

$$\omega = \sqrt{\frac{|\vec{p}||\vec{E}|}{I}} = \sqrt{\frac{pE}{2ma^2}} = \frac{\sqrt{2qaE}}{2ma^2} = \sqrt{\frac{qE}{ma}} = 2\pi f \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{qE}{ma}}$$

As before, the energy spectrum of the oscillating dipole would be quantized and thus:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right); n = 1,2,3,\dots; \omega = \sqrt{\frac{qE}{ma}}$$

Here, you also see Planck's constant which is given by:

$$\hbar = \frac{h}{2\pi} = 1.0546\text{x}10^{-34} \text{ J·s}$$

This is yet one more example of where concepts from the first semester are very important in the second semester of physics for a more complete picture.

Incidentally, you'll also need to know something about the electric polarization. The electric polarization of a material $\vec{P}$ is defined as the dipole moment per unit volume of the material. This can be difficult to calculate but it is a vector quantity.

You can also calculate the work required to orient a dipole from some angle θ (as I have defined it above) to the x-axis to some angle (where θ=0).
$$W = -pE\sin(\theta)$$
A bit of calculus magic happened here.

Since
$$\gamma = \theta - 90^0 \Rightarrow \theta = \gamma + 90^0 \Rightarrow \sin(\theta) = \sin(\gamma + 90^0) = \sin\gamma\cos(90^0) + \sin(90^0)\cos\gamma = \cos(\gamma)$$
we can rewrite this result in terms of the dot product. Thus, in terms of the angle between $\vec{E}$ and $\vec{p}$, we have:
$$U = -\vec{p}\cdot\vec{E}$$

Which would correspond to the energy of a dipole in an external electric field. This is important classically for a lot of dipoles in an external electric field. You can do an average over angles using Boltzman statistics to obtain an average angle (this leads to an equation known as the Langevin equation). Note that the negative sign insures that when the dipole moment is anti-aligned with the electric field, the energy is at a maximum.