

9.3 Electromagnetic waves in matter

Previously we obtained: for charge-free and current-free regions:

$$\vec{\nabla} \cdot \vec{D} = 0 \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

And for linear media: $\vec{D} = \epsilon \vec{E}$; $\vec{B} = \mu \vec{H}$

Provided that ϵ and μ have **no spatial dependence**, we can write these as:

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{B} = \epsilon \mu \frac{\partial \vec{E}}{\partial t}$$

These will also describe waves with a wave speed given by $v = \frac{1}{\sqrt{\epsilon \mu}}$.

We can relate this to the index of refraction (n) of the material:

$$n \equiv \frac{c}{v} \Rightarrow n = \frac{c}{v} = \frac{\sqrt{\epsilon_0 \mu_0}}{\sqrt{\epsilon \mu}}$$

If the material also has $\mu \approx \mu_0$, we can say:

$$n \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r}, \text{ the dielectric constant.}$$

Very quickly, all the previous results translate here with a replacement:

$$\vec{S} = \frac{1}{\mu} (\vec{E} \times \vec{B}) : u_{EM} = \frac{1}{2} \left(\epsilon E^2 + \frac{1}{\mu} B^2 \right) : I = \frac{1}{2} \epsilon \mu E_0^2$$

Following your author, we want to investigate what happens when a TEM wave (light) encounters an interface between two transparent materials.

From the previously obtained boundary conditions we have:

$$\begin{aligned} \epsilon_1 E_1^\perp &= \epsilon_2 E_2^\perp & E_1^\parallel &= E_2^\parallel \\ B_1^\perp &= B_2^\perp & \frac{1}{\mu_1} B_1^\parallel &= \frac{1}{\mu_2} B_2^\parallel \end{aligned}$$

Reflection and Transmission at normal incidence

The model is that in the x-y plane exists an infinite interface. The incident wave propagates in the +z direction. Let the electric field be polarized along the +x direction.

The **incident fields** are given by:

$$\begin{aligned}\vec{\tilde{E}}_I &= \tilde{E}_{0,I} e^{i(k_1 z - \omega t)} \hat{x} \\ \vec{\tilde{B}}_I &= \frac{1}{v_1} \tilde{E}_{0,I} e^{i(k_1 z - \omega t)} \hat{y}\end{aligned}$$

The **reflected fields** are given by:

$$\begin{aligned}\vec{\tilde{E}}_R &= \tilde{E}_{0,R} e^{i(-k_1 z - \omega t)} \hat{x} \\ \vec{\tilde{B}}_R &= -\frac{1}{v_1} \tilde{E}_{0,R} e^{i(-k_1 z - \omega t)} \hat{y}\end{aligned} \quad \text{where the (-) is required since } \vec{B} = \frac{1}{v} \hat{k} \times \vec{E}$$

The **transmitted fields** are given by:

$$\begin{aligned}\vec{\tilde{E}}_T &= \tilde{E}_{0,T} e^{i(k_2 z - \omega t)} \hat{x} \\ \vec{\tilde{B}}_T &= \frac{1}{v_2} \tilde{E}_{0,T} e^{i(k_2 z - \omega t)} \hat{y}\end{aligned}$$

At the interface, the amplitudes must join so that the boundary conditions are obeyed. Since we here have only normal incidence, there are no perpendicular components to consider. The boundary conditions were:

$$\begin{aligned}\epsilon_1 E_1^\perp &= \epsilon_2 E_2^\perp & E_1^\parallel &= E_2^\parallel \\ B_1^\perp &= B_2^\perp & \frac{1}{\mu_1} B_1^\parallel &= \frac{1}{\mu_2} B_2^\parallel\end{aligned}$$

Thus the requirement is:

$$\begin{aligned}\tilde{E}_{0,I} + \tilde{E}_{0,R} &= \tilde{E}_{0,T} \\ \frac{1}{\mu_1} \frac{1}{v_1} \tilde{E}_{0,I} - \frac{1}{\mu_1} \frac{1}{v_1} \tilde{E}_{0,R} &= \frac{1}{\mu_2} \frac{1}{v_2} \tilde{E}_{0,T}\end{aligned}$$

In accord with your author, define the following: $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}$

Then the two conditions that must be satisfied are:

$$\tilde{E}_{0,I} + \tilde{E}_{0,R} = \tilde{E}_{0,T} \quad \tilde{E}_{0,I} - \tilde{E}_{0,R} = \beta \tilde{E}_{0,T}$$

Solve for the reflected and transmitted amplitudes:

$$\tilde{E}_{0,R} = \left(\frac{1-\beta}{1+\beta} \right) \tilde{E}_{0,I}; \quad \tilde{E}_{0,T} = \left(\frac{2}{1+\beta} \right) \tilde{E}_{0,I}$$

Some implications of the results above:

Recall:

If the material also has $\mu \approx \mu_0$, we can say:

$$n \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r}, \quad \text{the dielectric constant.}$$

Then we can also say

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} \approx \frac{v_1}{v_2}$$

With this, the amplitudes become:

$$\tilde{E}_{0,R} = \left(\frac{1 - \frac{v_1}{v_2}}{1 + \frac{v_1}{v_2}} \right) \tilde{E}_{0,I} : \tilde{E}_{0,T} = \left(\frac{2}{1 + \frac{v_1}{v_2}} \right) \tilde{E}_{0,I}$$

$$\tilde{E}_{0,R} = \left(\frac{v_2 - v_1}{v_2 + v_1} \right) \tilde{E}_{0,I} : \tilde{E}_{0,T} = \left(\frac{2v_2}{v_2 + v_1} \right) \tilde{E}_{0,I}$$

$$\tilde{E}_{0,R} = \left(\frac{n_1 - n_2}{n_1 + n_2} \right) \tilde{E}_{0,I} : \tilde{E}_{0,T} = \left(\frac{2n_1}{n_1 + n_2} \right) \tilde{E}_{0,I}$$

You may have wondered, and now you see: why does the phase flip upon reflection by 180 degrees if $n_1 < n_2$. It is also, as you can see, not a completely simple thing to obtain.

We can also now find out details about energy transmitted and reflected. The time average Poynting vector for the incident wave is given by:

$$\langle \vec{S} \rangle = \frac{1}{2} v \epsilon E_{0,I}^2 \hat{z}$$

So the incident intensity is used to obtain the Reflection and Transmission coefficients:

$$I = \frac{1}{2} v \epsilon E_{0,I}^2 : R \equiv \left(\frac{E_{0,R}}{E_{0,I}} \right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 : T \equiv \left(\frac{E_{0,T}}{E_{0,I}} \right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

These coefficients obey $R+T=1$. The higher the transmission coefficient, the more energy will be passed through the interface. You can try this out for several indices of refraction for yourself. Note that if $n_1 = n_2$, all the energy is transmitted.