

Chapter 8: Conservation laws

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We have previously from Maxwell's equations obtained the continuity equation:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$$

For a electric and magnetic fields we have obtained:

$$U_E = \frac{\epsilon_0}{2} \iiint (\vec{E})^2 d^3r : U_M = \frac{1}{2\mu_0} \iiint (\vec{B})^2 d^3r$$

Thus the total energy stored in and electric and magnetic field is:

$$U_{EM} = \frac{1}{2} \iiint \left[\epsilon_0 (\vec{E})^2 + \frac{1}{\mu_0} (\vec{B})^2 \right] d^3r$$

This is derivable in another way so we'll follow your author here:

Suppose you have a charge in the presence of and electric field and a magnetic field at a time t . In a later instant, how much work is done by the charges moving around?

Starting with the Lorentz force law, for a particle initially at rest: (so no magnetic force):

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) : \text{Power} = \vec{F} \cdot \vec{v} = \frac{\vec{F} \cdot d\vec{L}}{dt} = q \vec{E} \cdot \vec{v}$$
$$\vec{F} \cdot d\vec{L} = q \vec{E} \cdot \vec{v} dt$$

$$\text{but } q = \iiint \rho d^3r : \vec{J} = \rho \vec{v}$$

So the total work done per unit time (the power) is given by:

$$\text{Power} = \frac{dW}{dt} = \iiint \rho \vec{E} \cdot \vec{v} d^3r = \iiint \vec{E} \cdot \vec{J} d^3r$$

But Ampere's law says:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\text{So } \vec{E} \cdot \vec{J} = \frac{1}{\mu_0} \vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

Now use the product rule:

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B}) \Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$\text{with Faraday's law: } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{To give: } \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = -\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \cdot (\vec{E} \times \vec{B}) = -\frac{1}{2} \frac{\partial B^2}{\partial t} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

So:

$$\text{repeat: Power} = \frac{dW}{dt} = \iiint \vec{E} \cdot \vec{J} d^3r$$

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_0} \vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = -\frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E} \times \vec{B}) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

$$\text{Also note: } \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{\partial B^2}{\partial t} \quad \text{and} \quad \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \frac{\partial E^2}{\partial t}$$

$$\frac{dW}{dt} = \iiint \left[-\frac{1}{2\mu_0} \frac{\partial B^2}{\partial t} - \frac{\epsilon_0}{2} \frac{\partial E^2}{\partial t} - \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \right] d^3r$$

$$\text{By Gauss's theorem: } \iiint \vec{\nabla} \cdot (\vec{E} \times \vec{B}) d^3r = \iint (\vec{E} \times \vec{B}) \cdot d\vec{A}$$

$$\text{We define the Poynting vector as: } \vec{S} \equiv \frac{1}{\mu_0} (\vec{E} \times \vec{B})$$

External work done on the system results in a change in the EM energy of the system and well something else that we're going to show is related to the Poynting vector.

Note the units of the Poynting vector:

$$\begin{aligned} [\mu_0] &= [B] \frac{[\text{Length}]}{[\text{Amps}]} \cdot \frac{1}{[\mu_0]} [B] = \frac{[\text{Amps}]}{[\text{Length}]} : [E] = \frac{[\text{Force}]}{[\text{Coulombs}]} \\ [S] &= \frac{[\text{Force}]}{[\text{Coulombs}]} \frac{[\text{Coulombs}]}{[\text{Seconds}][\text{Length}]} = [\text{Force}] \frac{[\text{Length}]}{[\text{Seconds}][\text{Length}]^2} = \frac{[\text{Power}]}{[\text{Area}]} \end{aligned}$$

Note: this is almost an intensity which is given by: $I = \langle \vec{S} \rangle \cdot \hat{n}$

$$\text{Where } u_{EM} = \frac{1}{2\mu_0} \vec{B} \cdot \vec{B} + \frac{1}{2} \epsilon_0 \vec{E} \cdot \vec{E} : u_{EM} = \iiint u_{EM} d^3r \quad u_{EM} \text{ is the EM energy density.}$$

$$\frac{dW}{dt} = \iiint \left[-\frac{1}{2\mu_0} \frac{\partial B^2}{\partial t} - \frac{\epsilon_0}{2} \frac{\partial E^2}{\partial t} - \vec{\nabla} \cdot \vec{S} \right] d^3r$$

Now consider a closed system in which no external work is done. Then:

$$\frac{\partial u_{EM}}{\partial t} = -\vec{\nabla} \cdot \vec{S}$$

This says then the time rate of energy change is equal to the negative divergence of the Poynting vector. This is it in differential form. In integral form:

$$\frac{\partial}{\partial t} \iiint u_{\text{EM}} d^3 r = \iiint \vec{\nabla} \cdot \vec{S} d^3 r \Rightarrow \frac{\partial U_{\text{EM}}}{\partial t} = - \oint \vec{S} \cdot d\vec{A}$$

This in words is the time rate of change of the EM energy is equal to the flux of the Poynting vector through a closed surface containing the energy density that is changing. The Poynting vector is the flux of energy per unit time: J/(s m²).

Example 8.1

Your author considers a wire held at a potential difference V of length L and radius a . A current flows in the wire (in the z direction). The electric field in the wire is given by:

$$\vec{E} = \frac{V}{L} \hat{z}$$

By Ampere's law, at the surface of the wire, the magnetic field is given by:

$$\vec{B} = \frac{\mu_0 I}{2\pi a} \hat{\phi}$$

At the surface of the wire, the Poynting vector is given by:

$$\vec{S} = \frac{1}{\mu_0} \frac{V}{L} \frac{\mu_0 I}{2\pi a} (\hat{z} \times \hat{\phi})$$

The cross product is :

$$\hat{z} \times \hat{\phi} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} = \hat{x}(-\cos\phi) - \hat{y}(\sin\phi) = -\hat{s}$$

$$\vec{S} = -\frac{1}{\mu_0} \frac{V}{L} \frac{\mu_0 I}{2\pi a} \hat{s}$$

Integrate over the surface of the wire:

$$\oint \vec{S} \cdot d\vec{A} = \frac{-IV}{2\pi a L} \oint a d\phi dz = IV$$

(the normal to the area points outward, so that the negative cancels.

$$\text{But the power radiated is given by } P = \frac{\partial U_{EM}}{\partial t} = IV$$

Which is pretty much exactly what we obtained with the power radiated, for example by a resistor. But you'll note here we did not need to refer specifically to something like the wire had a resistance; although the deeper truth is that we did. In fact, because we can establish a constant potential difference between the ends of the wire, this would require the presence of a resistance, R .

This permits writing the power in the three forms:

$$P = IV = I^2 R = \frac{V^2}{R}$$