7.1.1 Ohm's law

It is observed that a current is the result of a force on a charge per unit charge:

$$\vec{J} = \sigma \vec{E}$$

Here σ is called the “conductivity” of the system and is not surface charge density. If the charges are in the presence of an external magnetic field, we also have:

$$\vec{J} = \sigma \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

which you recognize as the Lorentz force.

However, for relatively small velocities the magnetic effect may not be significant.

Example 7.1: material with a cross sectional area A, length L and conductivity σ. If a potential difference V exists across the ends, what is the current that flows.

$$\vec{J} = \sigma \vec{E} : I = \oiint \vec{J} \cdot d\vec{A}$$

Model as two infinite parallel plate capacitors:

$$E A' = \frac{\sigma_Q}{\epsilon_0} A' \Rightarrow \vec{E} = \frac{\sigma_Q}{\epsilon_0} \hat{z}$$

$$I = \oiint \vec{J} \cdot d\vec{A} = \sigma \frac{\sigma_Q}{\epsilon_0} A$$

$$V = -\int_{z=L}^{z=0} \vec{E} \cdot d\vec{z} = \frac{\sigma_Q}{\epsilon_0} L = \frac{I}{\sigma A} L \Rightarrow R = \frac{L}{\sigma A}$$

Example 7.2: coaxial cylinders, maintained at V, with conductivity σ. In a length L what is the current that flows?

$$\vec{J}=\sigma\vec{E}:I=\oiint\vec{J}\cdot d\vec{A}$$

$$\oiint\vec{E}\cdot d\vec{A}=\frac{Q_{enc}}{\epsilon_0}\Rightarrow E(L)(2\pi s)=\frac{\lambda L}{\epsilon_0}\Rightarrow\vec{E}=\frac{\lambda}{2\pi s\epsilon_0}\hat{s}$$

$$I=\oint\sigma\left(\frac{\lambda}{2\pi\epsilon_0 s}\right)2\pi s\,dL=\frac{\lambda\sigma L}{\epsilon_0}\Rightarrow\frac{\lambda}{\epsilon_0}=\frac{I}{\sigma L}$$

$$\vec{E}=-\vec{\nabla}V=-\left(\frac{dV}{ds}\right)\hat{s}\Rightarrow V=-\int_{r=a}^{r=b}\vec{E}\cdot d\vec{L}=\frac{\lambda}{2\pi\epsilon_0}\int_{r=a}^{r=b}\frac{ds}{s}=\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$$

$$V=\frac{I}{2\pi\sigma L}\ln\left(\frac{b}{a}\right)\Rightarrow R=\frac{\ln\left(\frac{b}{a}\right)}{2\pi\sigma L}$$

Suppose $\sigma=\frac{k}{s}$, steady currents, between 2 cylindrical shells.

problem 7.4: Consider two cylindrical shells , one of radius a, the other (larger) of radius s which ultimately will have the maximum radius b. The same current crosses both cylinders. This means:

$$\oiint_a\vec{J}\cdot d\vec{A}=\oiint_s\vec{J}\cdot d\vec{A}$$

$$J_a(2\pi La)=J_s(2\pi Ls)\Rightarrow J=J_a\frac{a}{s}$$

$$\vec{J}=\sigma\vec{E}\Rightarrow\vec{E}=\frac{J_a}{\sigma}\frac{a}{s}\hat{s}=\frac{J_a}{\left(\frac{k}{s}\right)}\frac{a}{s}\hat{s}=\frac{J_a a}{k}\hat{s}$$

notice that as required, $\vec{\nabla}\cdot\vec{E}\neq\frac{1}{\sigma}\vec{\nabla}\cdot\vec{J}$

$$I=\oiint\vec{J}\cdot d\vec{A}=J_a\frac{a}{s}(2\pi sL)=J_a(2\pi aL)\Rightarrow J_a a=\frac{I}{2\pi L}$$

$$V=-\oint\vec{E}\cdot d\vec{s}=\frac{J_a a}{k}(b-a)=I\frac{(b-a)}{2k\pi L}\Rightarrow R=\frac{(b-a)}{2k\pi L}$$

problem 7.1: two concentric spheres (r=a,r=b) separated with a material of σ. Find the resistance when the shells are at a different potential.

$$\oiint \vec{E}\cdot d\vec{A}=\frac{Q_{enc}}{\epsilon_0}\Rightarrow E(4\pi r^2)=\frac{\sigma_Q(4\pi a^2)}{\epsilon_0}\Rightarrow \vec{E}=\frac{\sigma_Q}{\epsilon_0}\frac{a^2}{r^2}\hat{r}\Rightarrow \vec{J}=\sigma\left(\frac{\sigma_Q}{\epsilon_0}\right)\frac{a^2}{r^2}\hat{r}$$

$$I=\oiint \vec{J}\cdot d\vec{A}=\sigma\left(\frac{\sigma_Q}{\epsilon_0}\right)a^2(4\pi)\int_{r=a}^{r=b}\frac{dr}{r}=\sigma\left(\frac{\sigma_Q}{\epsilon_0}\right)a^2(4\pi)\ln\left(\frac{b}{a}\right)\Rightarrow\left(\frac{\sigma_Q}{\epsilon_0}\right)a^2=\frac{I}{4\pi\sigma\ln\left(\frac{b}{a}\right)}$$

$$V=-\int_{r=a}^{b}\vec{E}\cdot d\vec{r}=\frac{\sigma_Q}{\epsilon_0}a^2\left(\frac{1}{a}-\frac{1}{b}\right)=\frac{I}{4\pi\sigma\ln\left(\frac{b}{a}\right)}\left(\frac{1}{a}-\frac{1}{b}\right)\Rightarrow R=\frac{\left(\frac{1}{a}-\frac{1}{b}\right)}{4\pi\sigma\ln\left(\frac{b}{a}\right)}$$