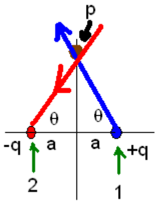
Physics 382: Electric Dipole Notes: From Physics 250 notes

(3) The electric dipole consists of a positive and a negative charge separated by a distance of 2a. Suppose in this case, your dipole had +q at x=a and -q at x=-a. Find an expression for the electric field along the y-axis. You should then be able to show that the electric field behaves as $\mathbf{E}_x \approx \mathbf{2kqa/y^3}$ at distant points along the y-axis.

We begin with the definition of the electric field:



$$\vec{E}_p = \sum_{i=1}^{n} k \frac{q_i}{\left|\vec{r}_p - \vec{r}_i\right|^2}\hat{r}_{ip}$$

Now we need to obtain the various vectors involved.

$$\vec{r}_1 = a\hat{x} : \vec{r}_2 = -a\hat{x} : \vec{r}_p = y_p\hat{y}$$

$$\vec{r}_{1p} = -a\hat{x} + y_p\hat{y} : \vec{r}_{2p} = a\hat{x} + y_p\hat{y}$$

$$\hat{r}_{1p} = \frac{-a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}} : \hat{r}_{2p} = \frac{a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}}$$

Now we need to use these in the definition of the electric field.

$$\vec{E}_p = k\frac{q_1}{\left|\vec{r}_{1p}\right|^2}\hat{r}_{1p} + k\frac{q_2}{\left|\vec{r}_{2p}\right|^2}\hat{r}_{2p}$$

We thus have: (letting q_1 be the same magnitude as q_2):

$$\vec{E}_p = k\frac{q}{a^2+y_p^2}\frac{-a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}} - k\frac{q}{a^2+y_p^2}\frac{a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}} = \frac{kq}{\left[a^2+y_p^2\right]^{3/2}}\left\{(-a-a)\hat{x} + (y_p - y_p)\hat{y}\right\} \Rightarrow \vec{E}_p = -\frac{2akq}{\left[a^2+y_p^2\right]^{3/2}}\hat{x}$$

This is the actual answer. Now let's look at how this behaves for y>>a. The binomial expansion says:

$$\left(a^2+y_p^2\right)^{3/2} = y_p^3\left(\left(\frac{a}{y_p}\right)^2+1\right)^{3/2} \approx y_p^3\left(1+\frac{3}{2}\left(\frac{a}{y_p}\right)^2+\dots\right)$$

Thus to first order, (if y>>a) we have:

$$\left(a^2+y_p^2\right)^{3/2} \approx \left|y_p\right|^3$$

The electric field at large distances along the perpendicular bisector of the dipole is:

$$\vec{E}_p \approx \frac{-2akq}{\left|y_p\right|^3}\hat{x}$$

Both of these results are extremely important for systems involving electric dipoles! It is also indeed very interesting to see that the dipole falls off as $1/y^3$ at large distances. The term p=2qa is called the magnitude of the electric dipole moment. We calculate the dipole moment for 2 equal and opposite charges as:

$$\vec{p} = \sum_{j=1}^{2} q_j\vec{r}_j = qa\hat{x} - q(-a\hat{x}) = q(2a\hat{x}) = q\vec{d}$$

where $\vec{d}$ is the vector pointing ***from the negative charge towards the positive charge***.

For a continuous charge distribution, the electric dipole moment is calculated as:

$$\vec{p} = \int_{\text{all charges}} \vec{r}_i\,\rho(\vec{r}_i)\,d^3r_i$$

In terms of the electric dipole moment at large distances along the symmetry axis we have (using only most significant terms):

$$\vec{E}_p = -k\frac{\vec{p}}{\left|y_p\right|^3}.$$

The following shows how this conclusion is obtained (this is for interested students):

Suppose you are off the symmetry axis:

$$\vec{r}_p = x_p\hat{x} + y_p\hat{y}$$

Then the calculation is a little bit more complicated: I am showing you the steps for future reference (the electric physical dipole is quite important for chemists).

$$\vec{r}_1 = a\hat{x} : \vec{r}_2 = -a\hat{x} : \vec{r}_p = x_p\hat{x} + y_p\hat{y}$$

$$\vec{r}_{1p} = \left(x_p - a\right)\hat{x} + y_p\hat{y} : \vec{r}_{2p} = \left(x_p + a\right)\hat{x} + y_p\hat{y}$$

$$\hat{r}_{1p} = \frac{\left(x_p - a\right)\hat{x} + y_p\hat{y}}{\sqrt{\left(x_p - a\right)^2 + y_p^2}} : \hat{r}_{2p} = \frac{\left(x_p + a\right)\hat{x} + y_p\hat{y}}{\sqrt{\left(x_p + a\right)^2 + y_p^2}}$$

The electric field is then:

$$\vec{E}_p = k\frac{q_1}{\left|\vec{r}_{1p}\right|^2}\hat{r}_{1p} + k\frac{q_2}{\left|\vec{r}_{2p}\right|^2}\hat{r}_{2p}$$

So:

$$\vec{E}_p = kq\left[\frac{\left(x_p - a\right)\hat{x} + y_p\hat{y}}{\left[\left(x_p - a\right)^2 + y_p^2\right]^{3/2}} - \frac{\left(x_p + a\right)\hat{x} + y_p\hat{y}}{\left[\left(x_p + a\right)^2 + y_p^2\right]^{3/2}}\right]$$

In the following, let $\vec{a} \equiv a\hat{x}$. Then:

$$\left[\left(x_p \pm a\right)^2 + y_p^2\right] = \left[x_p^2 + a^2 \pm 2x_p a + y_p^2\right] = \left[\vec{r}_p^{\,2} + \vec{a}^2 \pm 2\vec{r}_p \bullet \vec{a}\right] = \left[\vec{r}_p^{\,2} + \vec{a}^2 \pm 2\left|\vec{r}_p\right|\left|\vec{a}\right|\cos\left(\theta\right)\right]$$

(Here, the angle is with respect to the +x axis).

let's look at the approximations for r>>a.

$$\left[\vec{r}_p^{\,2} + \vec{a}^2 \pm 2\left|\vec{r}_p\right|\left|\vec{a}\right|\cos\left(\theta\right)\right]^{-3/2} = \left|\vec{r}_p^{\,2}\right|^{-3/2}\left[1 + \frac{\vec{a}^2 \pm 2\left|\vec{r}_p\right|\left|\vec{a}\right|\cos(\theta)}{\vec{r}_p^{\,2}}\right]^{-3/2} \approx \left|\vec{r}_p\right|^{-3}\left[1 \mp 3\frac{\left|\vec{r}_p\right|\left|\vec{a}\right|\cos(\theta)}{\vec{r}_p^{\,2}}\right] \approx \left|\vec{r}_p\right|^{-3}\left[1 \mp 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right]$$

So in the expression for E above, we'll replace:

$$\frac{1}{\left[\left(x_p \pm a\right)^2 + y_p^2\right]^{3/2}} \approx \frac{1}{\left|\vec{r}_p\right|^3}\left[1 \mp 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right]$$

We then have:

$$\vec{E}_p \approx \frac{kq}{\left|\vec{r}_p\right|^3}\left[\left[\left(x_p - a\right)\hat{x} + y_p\hat{y}\right]\left[1 + 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right] - \left[\left(x_p + a\right)\hat{x} + y_p\hat{y}\right]\left[1 - 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right]\right]$$

Simplifying:

$$\vec{E}_p \approx \frac{kq}{\left|\vec{r}_p\right|^3}\begin{bmatrix} x_p\hat{x}\left[+\left[1 + 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right] - \left[1 - 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right]\right] \\ +a\hat{x}\left[-\left[1 + 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right] - \left[1 - 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right]\right] \\ +y_p\hat{y}\left[\left[1 + 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right] - \left[1 - 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right]\right] \end{bmatrix} = \frac{kq}{\left|\vec{r}_p\right|^3}\begin{bmatrix} x_p\hat{x}\left[6\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right] \\ +a\hat{x}\left[-2\right] \\ +y_p\hat{y}\left[6\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right] \end{bmatrix} = \frac{kq}{\left|\vec{r}_p\right|^3}\left[-2a\hat{x} + 6\vec{r}_p\left[\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^{\,2}}\right]\right]$$

In terms of the dipole moment, we then have:

$$\vec{E}_p \approx \frac{k}{\left|\vec{r}_p\right|^3}\left[-\vec{p} + 3\vec{r}_p\left[\frac{\vec{r}_p \bullet \vec{p}}{\vec{r}_p^{\,2}}\right]\right] = 3\frac{k\hat{r}_p\left(\hat{r}_p \bullet \vec{p}\right)}{\left|\vec{r}_p\right|^3} - \frac{k\vec{p}}{\left|\vec{r}_p\right|^3}$$

Where in this case the (physical) electric dipole is $\vec{p} = 2qa\hat{x}$.

(4) Suppose in this case, your dipole had +q at x=a and -q at x=-a. Find an expression for the electric field along *the x-axis* at x>a. You should then be able to show that the electric field behaves as $\mathbf{E_x} \approx \mathbf{4kqa/x^3}$ at distant points along the *x* -axis. Then write the result in terms of the dipole moment.

Here, it's clear that the y-component of the resultant electric field vanishes. It is particularly easy to find the electric field in this case through direct application of the definition of the electric field.

$$\vec{E}_p = \sum_{i=1}^{n} k\frac{q_i}{|\vec{r}_p-\vec{r}_i|^2}\hat{r}_{ip} = k\frac{q_1}{(\vec{r}_p-\vec{r}_1)^2}\hat{r}_{1p} + k\frac{q_2}{(\vec{r}_p-\vec{r}_2)^2}\hat{r}_{2p}$$

We need to get each of the vectors. Also, let's assume for simplicity that we're along the +x axis here. Thus, at a point x_p along the +x axis, we have:

$\hat{r}_{1p} = +\hat{x} = \hat{r}_{2p}$

and $\vec{r}_p = +x_p\hat{x},\ \vec{r}_1 = +a\hat{x}, \vec{r}_2 = -a\hat{x}$. Let's find the distances:

$$|\vec{r}_p - \vec{r}_1| = |x_p - a| \Rightarrow |\vec{r}_p - \vec{r}_1|^2 = (x_p - a)^2$$
$$|\vec{r}_p - \vec{r}_2| = |x_p + a| \Rightarrow |\vec{r}_p - \vec{r}_2|^2 = (x_p + a)^2$$

so we then have:

$$E_p = k\frac{+q}{(x_p-a)^2} + k\frac{-q}{(x_p+a)^2} = kq\left(\frac{1}{x_p^2-2ax_p+a^2} - \frac{1}{x_p^2+2ax_p+a^2}\right) =$$
$$= kq\left(\frac{x_p^2+2ax_p+a^2-x_p^2+2ax_p-a^2}{(x_p^2+a^2)^2-4a^2x_p^2}\right) = kq\left(\frac{4ax_p}{x_p^4-2a^2x_p^2+a^4}\right) = 4kqa\left(\frac{x_p}{(x_p^2-a^2)^2}\right)$$

As x_p gets large, the only really important term of the denominator is x_p. Thus: $\vec{E} \approx \frac{4kqa}{|x_p|^3}\hat{x}$ in the +x region of space. In the –x region of space, the electric field is given by $\vec{E} \approx -\frac{4kqa}{|x_p|^3}\hat{x}$ In terms of the electric dipole defined above, we then have at large distances the electric field is given by (along the +x-axis):

$$\vec{E}_p \approx 2k\frac{\vec{p}}{x_p^3}$$

If you use our general result, you should obtain the same (approximate) result:

$$\vec{E}_p \approx \frac{k}{|\vec{r}_p|^3}\left[-\vec{p}+3\vec{r}_p\left[\frac{\vec{r}_p\bullet\vec{p}}{\vec{r}_p^2}\right]\right] = 3\frac{k\hat{r}_p(\hat{r}_p\bullet\vec{p})}{|\vec{r}_p|^3} - \frac{k\vec{p}}{|\vec{r}_p|^3} = 3\frac{k\vec{p}}{|\vec{r}_p|^3} - \frac{k\vec{p}}{|\vec{r}_p|^3} = 2\frac{k\vec{p}}{|\vec{r}_p|^3} = 2\frac{k\vec{p}}{x_p^3}$$

Remember, however, our expression for the dipole:

$$\vec{E}_p \approx \frac{k}{|\vec{r}_p|^3}\left[-\vec{p}+3\vec{r}_p\left[\frac{\vec{r}_p\bullet\vec{p}}{\vec{r}_p^2}\right]\right] = 3\frac{k\hat{r}_p(\hat{r}_p\bullet\vec{p})}{|\vec{r}_p|^3} - \frac{k\vec{p}}{|\vec{r}_p|^3}$$

is really only valid for r>>a whereas doing the exact calculation is always valid (since it is without approximation). This means that you can not always start with the field for the dipole to represent any dipole you run into! However at those times when you are in the correct region for approximation, it is appropriate to use this result.