2.4.2 Energy in a charge distribution

The work to establish a charge distribution (but not the charges themselves) is

$$W=\frac{1}{2}\sum_{i=1}^{i=n} q_i \sum_{\substack{j=1 \\ j\neq i}}^{j=n} \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q_j}{|r_{ij}|} \quad \text{(eq. 2.41 and 1/2)}$$

Which is the potential from all charges except the ith charge. Thus:

$$W=\frac{1}{2}\sum_{i=1}^{i=n} q_i V(\vec{r}_i) \quad \text{(eq 2.42)}$$

Where really the potential here is due to everything except the ith charge and is at the location of the ith charge.
You may remember in Physics 250 that I show this as a method to find this energy

$$W=\frac{1}{2}\begin{bmatrix} [+]^{j\rightarrow}/_{i\downarrow} & 1 & 2 & 3 & 4 & \dots & N \\ 1 & 0 & W_{12} & W_{13} & W_{14} & \dots & W_{1N} \\ 2 & W_{21} & 0 & W_{23} & W_{24} & \dots & W_{2N} \\ 3 & W_{31} & W_{32} & 0 & W_{34} & \dots & W_{3N} \\ 4 & W_{41} & W_{42} & W_{43} & 0 & \dots & W_{4N} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ N & W_{N1} & W_{N2} & W_{N3} & W_{N4} & \dots & 0 \end{bmatrix}$$

You fill out the matrix, noting that $W_{ij}=W_{ji}$ as:

$$W=\begin{bmatrix} [+]^{j\rightarrow}/_{i\downarrow} & 1 & 2 & 3 & 4 & \dots & N \\ 1 & 0 & W_{12} & W_{13} & W_{14} & \dots & W_{1N} \\ 2 & 0 & 0 & W_{23} & W_{24} & \dots & W_{2N} \\ 3 & 0 & 0 & 0 & W_{34} & \dots & W_{3N} \\ 4 & 0 & 0 & 0 & 0 & \dots & W_{4N} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ N & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

where each term is given by:

$$W_{jk}=q_j V(\text{at } r_j \text{ due to charge } q_k) \; .$$

Again, what this represents is the work to establish a charge distribution given that individual charges already exist. It answers the question how much work is required to put this distribution of charges together starting with the fact that the charges exist.

2.4.3 The continuous distribution.

3 forms for this are:

$$W=\frac{1}{2}\underset{\text{all } q_j}{\oiiint}\rho_j V d^3 r : W=\frac{1}{2}\underset{\text{all } q_j}{\oiint}\sigma_j V dA : W=\frac{1}{2}\underset{\text{all } q_j}{\oint}\lambda_j V ds$$

V here is the entire potential and, in fact, it is the potential located at the charge coordinates because it is the potential at the position of the charge distribution that is used to determine the energy. Note the discussion by your author on page 95, especially at the bottom where he says quite the opposite of what I said here and so we better stick with him. It's not that I am particularly comfortable with this distinction at all.

The first form reads:

$$W=\frac{1}{2}\underset{\text{all } q_i}{\oiiint}\rho_i V\vec{r}_i d^3 r_i$$

Now from Gauss's law:

$\rho=\epsilon_0\vec{\nabla}\cdot\vec{E}$ which I always remember as $\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0}$.

This permits us to write the work as:

$$W=\frac{1}{2}\oiiint\epsilon_0\vec{\nabla}\cdot\vec{E}V d^3 r$$

Following your author, you integrate by parts to transfer the derivative to V:
Remember by parts:

$$\int_a^b\frac{d}{dx}(fg)dx=fg|_a^b=\int_a^b f\left(\frac{dg}{dx}\right)dx+\int_a^b g\left(\frac{df}{dx}\right)dx$$

$$W=\frac{1}{2}\epsilon_0\left(-\oiiint\vec{E}\cdot\vec{\nabla}V d^3 r+\oiint V\vec{E}\cdot d\vec{A}\right)$$

Since $\vec{E}=-\vec{\nabla}V\Rightarrow W=\frac{1}{2}\epsilon_0\left(\oiiint\vec{E}\cdot\vec{E}d^3 r+\oiint V\vec{E}\cdot d\vec{A}\right)$

Now extend these integrals to all space (since they are actually over all charges, there is nothing wrong with integrating out there where the charge density is zero). The second of these integrals goes to zero based upon my argument which is for a normal physical charge distribution, E and V approach zero at infinity. Following your author's argument actually works also which is that the entire integral approaches zero as 1/r. We then have:

$$W=\frac{1}{2}\epsilon_0\underset{\text{all space}}{\oiiint}E^2 d^3 r=\underset{\text{all space}}{\oiiint}u_e d^3 r$$

Example 2.8 modified

Find the energy of a uniformly charged spherical conducting sphere of radius a.

$$W=\frac{\epsilon_0}{2}\oiiint E^2 d^3 r$$

We can find the electric field outside the sphere from Gauss's law:

$$\Phi_E=\frac{Q_{enc}}{\epsilon_0}\Rightarrow E(4\pi r^2)=\frac{\sigma(4\pi a^2)}{\epsilon_0}\Rightarrow \vec{E}=\frac{\sigma a^2}{\epsilon_0 r^2}\hat{r}=\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

Inside the sphere, since it is a conducting simply enclosed boundary, the electric field is zero (and the potential is a constant). So, the energy is:

$$W=\frac{\epsilon_0}{2}\oiiint\frac{Q^2}{16\pi^2\epsilon_0^2 r^4}r^2 dr\sin\theta d\theta d\phi$$

$$W=\frac{\epsilon_0}{2}\left(\frac{Q^2}{4\pi\epsilon_0^2}\right)\int_{r=a}^{r=\infty}\frac{dr}{r^2}=\frac{Q^2}{8\pi\epsilon_0}\left(\frac{-1}{r}\right)_a^\infty=\frac{Q^2}{8\pi\epsilon_0 a}$$

Superposition of energy

Suppose the electric field comes from 2 parts; what is the energy?

$$W=\frac{1}{2}\epsilon_0\oiiint E^2 d^3 r=\frac{1}{2}\epsilon_0\oiiint\left(\vec{E}_1+\vec{E}_2\right)^2 d^3 r=W_1+W_2+\epsilon_0\oiiint\left(\vec{E}_1\cdot\vec{E}_2\right)d^3 r$$

In particular, your author points out, doubling charge everywhere implies a quadrupling of the energy. Yes this makes you also uncomfortable in certain circumstances (the arguments for the parallel plate capacitor, for example). I suggest you need to be extremely careful here: Potentials and electric fields superimpose but the same can not be said for energies. They sure look awfully similar but the correct way to calculate the energy of a charge distribution is to find the total electric field of the distribution and take it from there. It is, in fact, that cross term above that says clearly that the energies do not superimpose simply by finding the energies associated with the individual electric fields and adding them up:

$$\epsilon_0\oiiint\left(\vec{E}_1\cdot\vec{E}_2\right)d^3 r\quad .$$

Problem 2.34

2 concentric spherical shells, radii a <b with +q on the inner shell and -q on the outer shell. Calculate the energy.

Essentially, we write, using E outside to be zero and E<a is zero:

$$W=\frac{\epsilon_0}{2}\oiiint\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 d^3 r=\frac{4\pi\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\int_a^b\frac{dr}{r^2}=\frac{-Q^2}{8\pi\epsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right)=\frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

Done the wrong way:

$$W=W_1+W_2=\frac{Q^2}{8\pi\epsilon_0 a}+\frac{Q^2}{8\pi\epsilon_0 b}=\frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{a}+\frac{1}{b}\right)$$

which is incorrect.

2.5 Conductors

(We are talking about electrostatics here, not dynamics)

A conductor is an equipotential surface. The electric field inside a conductor is zero; the electric field inside a simply closed region inside the conductor is zero. It take no work to move a charge along a conductor. These are the static rules.

I believe you should, at least for now, assume we have perfect conductors. In particular, if you have a thick slab of material placed in an otherwise uniform electric field (produced from external sources) then an induced charge distribution will develop across the slab of conductor to produce an electric field internal to the conductor which will exactly counteract the external electric field in the interior of the conductor. In a high symmetry situation, such as a rectangular slab of metal, this would be a surface charge density. However, if the external field were not uniform (say it arose from a point charge) then the result would be that the distribution of charges on the two sides of the metal slab would not necessarily be equal. Further, if the slab were finite, you would have additional charges along the edges.

All **excess charge** on a conductor must lie on the surface since there is no where else for it to be and it forms a surface charge density; which is really different from discrete charges. I would go so far as to say that a discrete charge can not be identified on the surface of a conductor because the conductor will respond to this by changing the surface charge density. However, and this is probably important, the surface charge density on a conductor is not necessarily constant. In particular, it will be higher at the edges.

At the surface of the conductor, any E which exists must be normal to the (local) conducting surface.

Holding a charge near a conductor will result in an attractive force because inside the conductor; a charge density will form in which the opposite charges are close to the external charge. This would not always be easy to calculate.

And finally a very clear statement about what if we hold a charge inside a simply connected void which is completely surrounded by conductor. The very clear statement is that there will be an electric field inside the void. However, the electric field will not penetrate the interior volume of the conductor. Never-the-less, a charge will form on an external surface of the conductor which has, as its origin, the internal charge held inside the void and thus; communicates the presence of the internal charge to the external world. This happens because of the requirement that the inner wall be equal and opposite to the charge held at the center: the charges are “pulled” from the surface during a short period of time which can not be characterized as “static”. **I suggest close reading on pages 99, 100, and 101.**