linear dielectrics
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for many materials, provided E is not too strong, the polarization is proportional to E:

$$\vec{P}=\epsilon_0\chi_E\vec{E}$$

where the susceptibility is χ_E .

There is a little problem here though: the electric field is due to free charges , externally applied fields and the bound charges that might be formed or are present. For this reason, it might be best to use D since it comes mostly only from free charges so long as the required symmetry is present.

$$\vec{D}=\epsilon_0\vec{E}+\vec{P}=\epsilon_0\vec{E}+\epsilon_0\chi_E\vec{E}=\epsilon_0\vec{E}\left(1+\chi_E\right)=\epsilon\vec{E};\epsilon\equiv\epsilon_0\left(1+\chi_E\right);\epsilon_r\equiv\frac{\epsilon}{\epsilon_0}=1+\chi_E$$

Example 4.5: Metal sphere of radius a, charge Q, surrounded by linear dielectric to radius b. Find the potential at the center.

Find out which method is correct somehow

Since the material is linear, $\vec{P}=\epsilon_0\chi_E\vec{E}$ with $\epsilon\equiv\epsilon_0\left(1+\chi_E\right)$.

D is given by:

$$\vec{D}=\epsilon_0\vec{E}+\vec{P}=\epsilon_0\left(1+\chi_E\right)\vec{E}\Rightarrow\vec{E}=\frac{\vec{D}}{\epsilon_0\left(1+\chi_E\right)}$$

If D arises solely from free charges, then:

$$\vec{D}=\frac{Q}{4\pi r^2}\hat{r}\Rightarrow\vec{E}=\frac{Q}{4\pi r^2\epsilon_0\left(1+\chi_E\right)}\hat{r}\Rightarrow\vec{P}=\frac{Q}{4\pi r^2}\frac{\chi_E}{\left(1+\chi_E\right)}\hat{r}\Rightarrow\sigma_{b,a}=\frac{-Q}{4\pi a^2}\frac{\chi_E}{1+\chi_E}$$

The value of the electric field at the surface at radius a is:

$$\Phi_E=\frac{Q_{enc}}{\epsilon_0}\Rightarrow\vec{E}=\frac{Q}{4\pi\epsilon_0 a^2}\hat{r}$$

Now the electric field at that point introduces a polarization. This results in a bound surface charge density given by:

$$\sigma_{b,a}=\hat{n}\cdot\vec{P}=-\hat{r}\cdot\epsilon_0\chi_E\vec{E}=-\chi_E\frac{Q}{4\pi a^2}$$

By symmetry this bound surface charge density does not change E for r<a.

We want to find the volume bound charge density now. This is given by:

$$\rho_b=-\vec{\nabla}\cdot\epsilon_0\chi_E\vec{E}=0$$

We now find the electric field for b>r>a

$$\vec{E}=\vec{E}_{from\ Q}+\vec{E}_{from\,\sigma_{b,a}}$$

$$\vec{E}_{from\ Q}=\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$\vec{E}_{from\ \sigma_{b,a}}=\frac{1}{4\pi\epsilon_0 r^2}\left[-\chi_E\frac{Q}{4\pi a^2}\left(4\pi a^2\right)\right]\hat{r}=-\chi_E\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$\vec{E}=\frac{Q}{4\pi\epsilon_0 r^2}\left[1-\chi_E\right]\hat{r}$$

Now let's find the induced charge at b:

again $\vec{P}=\epsilon_0\chi_E\vec{E}$ and $\sigma_{b,b}=\hat{r}\cdot\epsilon_0\chi_E\vec{E}$

So: $\sigma_{b,b}=\frac{Q}{4\pi b^2}\chi_E\left[1-\chi_E\right]$

We can now find E outside the sphere:

$$\vec{E}=\vec{E}_{\text{from }Q}+\vec{E}_{\text{from }\sigma_{b,a}}+\vec{E}_{\text{from }\sigma_{b,b}}$$

$$\vec{E}=\left[\frac{Q}{4\pi\epsilon_0 r^2}\left[1-\chi_E\right]+\frac{1}{4\pi\epsilon_0 r^2}\left[\frac{Q}{4\pi b^2}\chi_E\left(1-\chi_E\right)\left(4\pi b^2\right)\right]\right]\hat{r}=\frac{Q}{4\pi\epsilon_0 r^2}\left[1-\chi_E+\chi_E-\chi_E^2\right]\hat{r}=\frac{Q}{4\pi\epsilon_0 r^2}\left[1-\chi_E^2\right]\hat{r}$$

$$\oiint\vec{D}\cdot d\vec{A}=Q_f\Rightarrow\vec{D}=\frac{Q}{4\pi r^2}\hat{r}\Rightarrow r>b:\vec{E}=\frac{Q}{4\pi\epsilon_0 r^2}\hat{r};a<r<b:\vec{E}=\frac{Q}{4\pi\epsilon r^2}\hat{r};r<a:D=E=P=0$$

$$V=-\int_\infty^b\frac{Q}{4\pi\epsilon_0 r^2}dr-\int_b^a\frac{Q}{4\pi\epsilon r^2}dr=\frac{Q}{4\pi}\left[\frac{1}{\epsilon_0 b}-\frac{1}{\epsilon a}+\frac{1}{\epsilon b}\right]$$

Inside;

$$\vec{P}=\frac{\left(\epsilon-\epsilon_0\right)_E Q}{4\pi\epsilon_0 r^2}\hat{r}$$