2.2.1 : divergence of E and Gauss's law

Consider a point charge q_i located at the origin. Find the electric flux on a sphere surrounding the charge.

$$\oiint_{surface} \vec{E}\cdot d\vec{A} = \oiint k\frac{q_i}{r^2}\hat{r}\cdot r^2 \sin\theta\, d\theta\, d\phi\, \hat{r} = k q_i (4\pi) = \frac{1}{4\pi\epsilon_0} 4\pi q_i = \frac{q_i}{\epsilon_0}$$

If we have several charges then the field obeys:

$$\vec{E} = \sum_{n=1}^{N} \vec{E}_n = \sum_{n=1}^{N} \oiint_{surface} \vec{E}_n \cdot d\vec{A} = \sum_{n=1}^{N} \frac{q_i}{\epsilon_0} = \frac{Q_{enclosed}}{\epsilon_0}$$

Thus for any simply closed surface, we have:

$$\Phi_E = \oiint_{surface} \vec{E}\cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0}$$

This is Gauss's law in integral form (an integral equation).

The divergence theorem states:

$$\oiint_{surface} \vec{E}\cdot d\vec{A} = \oiiint_{volume} \left(\vec{\nabla}\cdot\vec{E}\right) d^3 r$$

Also we can find the enclosed charge by:

$$Q_{enc} = \oiiint_{volume} \rho\, d^3 r$$

If we put these together we then have:

$$\oiiint_{volume} \left(\vec{\nabla}\cdot\vec{E}\right) d^3 r = \frac{1}{\epsilon_0} \oiiint_{volume} \rho\, d^3 r \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

This is Gauss's law in differential form (a differential equation).

Note that the integral, and the divergence is with respect to space points.

2.2.2

Given a charge distribution, we can calculate E by Coulomb's law (the charge density is zero outside the region of interest

$$\vec{E}_p = k \oiiint_{all\ space} \frac{\rho(r_i)}{r_{ip}^2} \hat{r}_{ip}\, d^3 r_i$$

If we want the divergence of this, it is:

$$\vec{\nabla}_p \cdot \vec{E}_p = k \oiiint_{all\ space} \vec{\nabla}_p \cdot \left[\frac{\hat{r}_{ip}}{r_{ip}^2} \rho(r_i)\right] d^3 r_i$$

The differentiation here though is with respect to space points and not charge points. This permits:

$$\vec{\nabla}_p \cdot \left[\frac{\hat{r}_{ip}}{r_{ip}^2}\rho(r_i)\right] = \rho(r_i)\vec{\nabla}_p \cdot \left[\frac{\hat{r}_{ip}}{r_{ip}^2}\right] = \rho(r_i) 4\pi \delta^3(r_{ip}) : \text{see eq } 1.111$$

$$\vec{\nabla}_p \cdot \vec{E}_p = k\rho(r_i) = \frac{1}{4\pi\epsilon_0}\rho(r_i)$$

$$\text{since} \quad k = \frac{1}{4\pi\epsilon_0} \ .$$

This is again, Gauss's law in differential form.