

### The 3 D delta function

In 3-dimensions, the delta function would appear as:

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z) .$$

where  $\vec{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$  is the position vector. The 3-d delta function is defined by:

$$\delta^3(\vec{r}) = \begin{cases} 0 & \text{if } x,y,z \neq (0,0,0) \\ \infty & \text{if } (x,y,z) = (0,0,0) \end{cases} \text{ with}$$

$$\int_{\text{all space}} \delta^3(\vec{r}) d^3r = \int_{x=-\infty}^{x=+\infty} \int_{y=-\infty}^{y=+\infty} \int_{z=-\infty}^{z=+\infty} \delta(x)\delta(y)\delta(z) dx dy dz = 1 .$$

We also then have the shifting property:

$$\int_{\text{all space}} f(\vec{r}) \delta^3(\vec{r} - \vec{a}) d^3r = f(\vec{a}) .$$

In terms of the earlier divergence, we can now identify the 3-d delta function provided it satisfies what we found earlier:

$$4\pi \delta^3(\vec{r}) = \vec{\nabla} \cdot \left( \frac{\hat{r}}{r^2} \right) .$$

more generally, in terms of  $r_{ip}$  we have:

$$4\pi \delta^3(\vec{r}_{ip}) = \vec{\nabla}_p \cdot \left( \frac{\hat{r}_{ip}}{r_{ip}^2} \right)$$

where the divergence operator  $\vec{\nabla}_p$  is written so as to emphasize differentiation with respect to the p coordinates and  $\vec{r}_{ip} \equiv \vec{r}_p - \vec{r}_i$  .

It is probably important to emphasize here that you should not forget the "strength" factor of  $4\pi$  when you are working in 3-D space with the divergence which is easy to forget.

Something like this might work: "if you have 3, you need 4 when diverging."

To emphasize: the delta function is not:  $\delta^3(\vec{r}_{ip}) \neq \vec{\nabla}_p \cdot \left( \frac{\hat{r}_{ip}}{r_{ip}^2} \right) !!!$

I am going to also follow the examples from your author now.

Example 1.16: Evaluate

$$J \equiv \iiint (r^2 + 2) \vec{\nabla} \cdot \left( \frac{\hat{r}}{r^2} \right) d^3r$$

where the volume of interest is a sphere of radius R centered at the origin.

**Solution 1: I'll only say look at your author's solution 2.**

According to the definition of the 3-D Dirac function, we have:

$$J = \iiint (r^2 + 2) 4\pi \delta^3(\vec{r}) d^3r = 4\pi(0 + 2) = 8\pi .$$

### Problem 1.46

- (a) Write an expression for the electric charge density  $\rho(\vec{r})$  of a point charge .  
Make sure that the volume integral of  $\rho$  equals  $q$ .

This comes actually right from the definition of the 3-D delta function:

$$q \delta^3(\vec{r} - \vec{r}_i) = \rho(\vec{r}): \int_{\text{all charges}} Q \delta^3(\vec{r} - \vec{r}_i) d^3r_i = \int_{\text{all space}} Q \delta^3(\vec{r} - \vec{r}_i) d^3r = Q(\vec{r}_i) = Q$$

Remember: An important test of a charge density is does the integral over all charges give the total charge Q?

- (b) What is the charge density of an electric dipole, consisting of a point charge  $-q$  at the origin and a point charge  $+q$  at  $\vec{a}$  ?

$$\rho(\vec{r}) = -q \delta^3(\vec{r} - \vec{0}) + q \delta^3(\vec{r} - \vec{a})$$

What about if the negative charge were at  $-\vec{a}$  ?

$$\rho(\vec{r}) = -q \delta^3(\vec{r} + \vec{a}) + q \delta^3(\vec{r} - \vec{a})$$

- (c) What is the charge density of a uniform, infinitesimally thin shell of radius R and total charge Q, centered at the origin?

[ Be careful: the integral over all space must equal Q.]

Here, the basis of the answer is:

$$\rho(\vec{r}) = (\text{constant}) \times \delta^3(\vec{r} - \vec{R})$$

to evaluate the constant, we need to do the integral:

$$\iiint_{\text{all space}} \rho(\vec{r}) d^3r = \int_{r=0}^{r=\infty} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} C \delta^3(\vec{r} - \vec{R}) r^2 dr \sin\theta d\theta d\phi = 4\pi C R^2 = Q$$

$$\Rightarrow C = \frac{Q}{4\pi R^2} \Rightarrow \rho(\vec{r}) = \frac{Q}{4\pi R^2} \delta^3(\vec{r} - \vec{R})$$

Note: if you had N charges  $q_i$ , located at  $\vec{r}_i; i=1,2,3,\dots,N$ , the charge density is

$$\rho(\vec{r}) = \sum_{i=1}^{i=N} q_i \delta^3(\vec{r} - \vec{r}_i)$$

Also worth noting that the dimensions of the 3-D delta function are inverse volume. This is all important because I have now told you how in general to describe lots of charge distributions in a general way; something that is really useful.

Problem 1.48 Evaluate the integral

$$J \equiv \iiint e^{-r} \left( \nabla \cdot \frac{\hat{r}}{r^2} \right) d^3 r$$

where the volume is a sphere of radius R, centered at the origin (I'll do with the 3-D delta function, you do with parts). Reference Example 1.16.

$$\nabla \cdot \left( \frac{\hat{r}}{r^2} \right) = 4\pi \delta^3(\vec{r})$$

$$J \equiv \iiint e^{-r} 4\pi \delta^3(\vec{r} - \vec{0}) d^3 r = \int_{r=0}^{r=+\infty} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} e^{-r} 4\pi \delta^3(\vec{r} - \vec{0}) r^2 dr \sin\theta d\theta d\phi$$

so:

$$= e^0 \int_{\text{all space}} 4\pi \delta^3(r) d^3 r = 4\pi$$