

Supplementary problems for electrostatics

2020

1. Three identical charges each with $q=1\mu\text{C}$ are arranged at the vertices of an equilateral triangle with a side of length 1m . One vertex is at $(-1/2 \text{ m}, 0)$ and the second vertex is located at $(+1/2, 0)$ with the third vertex along the $+y$ axis at $(0, h)$. Find the electric field 1/2 way up the height of the triangle. **Revised 2020.**

2. A square has charges at the following locations and values:

$(1:1\mu, -1, -1), (2:2\mu, 1, -1), (3:3\mu, 1, 1), (4:4\mu, -1, 1)$.

Find the value of the vector electric field at the origin.

3. Three charges are arranged as follows:

$(1:1\mu, -1, 0), (2:-1\mu, 1, 0), (3:1\mu, 0, 1)$.

Find the vector electric field at the origin.

4. A sphere of radius a has a charge density which varies as $\rho = \rho_0 \frac{r}{a}$ inside and zero outside the sphere. Find the electric field inside and outside the sphere.

5. A parallel plate capacitor has a total charge $Q=1\mu\text{C}$ on one plate and $Q=-1\mu\text{C}$ on the other plate. The plates have a cross sectional area of 1 m^2 and are separated by a distance of 0.1 m. What is the value of the electric field near the center of the capacitor?

1. Three identical charges each with $q=1\mu\text{C}$ are arranged at the vertices of an equilateral triangle with a side of length 1m. One vertex is at $(-1/2, 0)$ and the second vertex is located at $(+1/2, 0)$ with the third vertex along the $+y$ axis at $(0, h)$. Find the electric field 1/2 way up the height of the triangle. **Revised 2020**

The hard part here is to locate coordinates of the point. Note that this is not the center of the triangle. Let the triangle have a side of length L . Locate one point at the origin. Locate the second point at $(L, 0)$. We need to find the coordinates of the center, and the 3rd point. The x coordinate of the 3rd point is $x=L/2$. The y coordinate is then given by:

$$L^2 = \left(\frac{L}{2}\right)^2 + y^2 \Rightarrow y^2 = L^2 - \frac{L^2}{4} = 3\frac{L^2}{4} \Rightarrow y = \frac{\sqrt{3}}{2}L.$$

The coordinate of the point is:

$$p = \left(\frac{L}{2}, \frac{\sqrt{3}}{4}L\right).$$

In our present case, $L=1$ m. We thus have the coordinates for this particular orientation:

$$1:(1\mu; -0.5, 0), 2:(1\mu; 0.5, 0), 3:(1; 0, \frac{\sqrt{3}}{2}), p:(0, \frac{\sqrt{3}}{4})$$

Calculate the following vectors: $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_p, \vec{r}_{1p}, \vec{r}_{2p}, \vec{r}_{3p}$. These vectors are:

$$\vec{r}_p = 0\hat{x} + \frac{\sqrt{3}}{4}\hat{y}$$

$$\vec{r}_1 = -.5\hat{x} + 0\hat{y}; \vec{r}_{1p} = \vec{r}_p - \vec{r}_1 = 0\hat{x} + \frac{\sqrt{3}}{4}\hat{y} - (-.5\hat{x} + 0\hat{y}) = 0.5\hat{x} + \frac{\sqrt{3}}{4}\hat{y}$$

$$\vec{r}_2 = 1\hat{x} + 0\hat{y}; \vec{r}_{2p} = \vec{r}_p - \vec{r}_2 = \left(0\hat{x} + \frac{\sqrt{3}}{4}\hat{y}\right) - (1\hat{x} + 0\hat{y}) = -1\hat{x} + \frac{\sqrt{3}}{4}\hat{y}$$

$$\vec{r}_3 = 0\hat{x} + \frac{\sqrt{3}}{2}\hat{y}; \vec{r}_{3p} = \vec{r}_p - \vec{r}_3 = \left(0\hat{x} + \frac{\sqrt{3}}{4}\hat{y}\right) - \left(0\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right) = 0\hat{x} - \frac{\sqrt{3}}{4}\hat{y}$$

Now write E directly.

$$\vec{E} = k_\mu \left[\frac{.5\hat{x} + \frac{\sqrt{3}}{4}\hat{y}}{\left[\frac{1}{4} + \frac{3}{16}\right]^{3/2}} + \frac{-.5\hat{x} + \frac{\sqrt{3}}{4}\hat{y}}{\left[\frac{1}{4} + \frac{3}{16}\right]^{3/2}} + \frac{0\hat{x} - \frac{\sqrt{3}}{4}\hat{y}}{\left[\frac{3}{16}\right]^{3/2}} \right] = 0\hat{x} + \frac{\sqrt{3}}{4} k_\mu \hat{y} \left[\frac{2}{\left[\frac{7}{16}\right]^{3/2}} - \frac{1}{\left[\frac{3}{16}\right]^{3/2}} \right]$$

$$\vec{E} = 0\hat{x} + k_\mu \hat{y} (1.496 - 5.333) = 0\hat{x} - 3.84 k_\mu = 0\hat{x} - 34521 \hat{y} \frac{\text{N}}{\text{C}}$$

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Find the value of the vector electric field at the origin.

Notice that if $\vec{r}_p = 0\hat{x} + 0\hat{y}$, then $\vec{r}_{ip} = \vec{r}_p - \vec{r}_i = \vec{0} - \vec{r}_i = -\vec{r}_i$. We can use this to write down the electric field directly:

$$\vec{E}_p = k_\mu \left[\frac{1(1\hat{x} + 1\hat{y})}{[1^2 + 1^2]^{3/2}} + \frac{2(-1\hat{x} + 1\hat{y})}{[2]^{3/2}} + \frac{3(-1\hat{x} - 1\hat{y})}{2\sqrt{2}} + \frac{4(\hat{x} - \hat{y})}{2\sqrt{2}} \right]$$

$$\Rightarrow \vec{E}_p = \frac{k_\mu}{2\sqrt{2}} [\hat{x}(1 - 2 - 3 + 4) + \hat{y}(1 + 2 - 3 - 4)] = \frac{k_\mu}{2\sqrt{2}} (-4\hat{y}) = \frac{-2k_\mu}{\sqrt{2}} \hat{y} = -\sqrt{2}k_\mu \hat{y} = -12713\hat{y} \frac{N}{C}$$

3. Three charges are arranged as follows: $(1:1\mu, -1, 0), (2:-1\mu, 1, 0), (3:1\mu, 0, 1)$.

Find the vector electric field at the origin.

This satisfies the condition $\vec{r}_p = \vec{0}$. We can then write the field directly.

$$\vec{E}_p = k_\mu \left[\frac{1(1\hat{x} + 0\hat{y})}{[1^2 + 0^2]^{3/2}} + \frac{-1(-1\hat{x} + 0\hat{y})}{1} + 1(0\hat{x} - 1\hat{y}) \right] = k_\mu [\hat{x}(1 + 1 + 0) + \hat{y}(0 + 0 - 1)] = k_\mu [2\hat{x} - 1\hat{y}]$$

$$\Rightarrow \vec{E}_p = [17980\hat{x} - 8990\hat{y}] \frac{N}{C}$$

4. A sphere of radius a has a charge density which varies as $\rho = \rho_0 \frac{r}{a}$ inside and zero outside the sphere. Find the electric field inside and outside the sphere.

One important point from this problem: you need to be able to integrate the charge density over a sphere. Here is the simplification if you have spherical symmetry, for a sphere which is located at the origin:

$$\iiint \rho(r) d^3r = 4\pi \int \rho(r)(r^2 dr)$$

(1) What is the total charge Q deposited on the sphere?

$$Q = 4\pi \int_{r=0}^{r=a} \rho(r) r^2 dr = 4\pi \frac{\rho_0}{a} \int_{r=0}^{r=a} r^3 dr = \pi \frac{\rho_0}{a} a^4 = \pi \rho_0 a^3 \Rightarrow \rho_0 = \frac{Q}{\pi a^3}$$

Outside the sphere: choose a Gaussian surface of radius R , centered on the sphere of charge. On this Gaussian Surface, $\vec{A} = A\hat{r}$; $|\vec{E}| = \text{constant}$: $\vec{E} = |\vec{E}|\hat{r}$.

Gauss's law says: $\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$

Calculate the flux:

$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = \oiint E dA = E \oiint dA = E(4\pi r^2): Q_{\text{enc}} = Q \Rightarrow \vec{E}_{\text{out}} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} = \frac{\rho_0 a^3}{4\epsilon_0 r^2} \hat{r}.$$

Inside the sphere: $Q_{\text{enc}} = 4\pi \int_{r=0}^{r=r} \rho r^2 dr = 4\pi \int_{r=0}^{r=r} \rho_0 \frac{r}{a} r^2 dr = \frac{4\pi\rho_0}{a} \int_{r=0}^{r=r} r^3 dr = \frac{4\pi\rho_0}{a} \frac{r^4}{4} = \pi \frac{\rho_0}{a} r^4.$

So from Gauss's law we have (using the same Gaussian surface but now $r \leq a$):

$$E_{\text{in}}(4\pi r^2) = \frac{\pi}{\epsilon_0} \frac{\rho_0}{a} r^4 \Rightarrow \vec{E}_{\text{in}} = \frac{\rho_0 r^2}{4\epsilon_0 a} \hat{r} = \frac{Q}{\pi a^3} \frac{r^2}{4\epsilon_0 a} \hat{r} = \frac{Q}{4\pi\epsilon_0 a^4} r^2 \hat{r}$$

Notice at the surface, the inside solution is equal to the outside solution.

5. A parallel plate capacitor has a total charge $Q = 1\mu\text{C}$ on one plate and $-Q = -1\mu\text{C}$ on the other plate. The plates have a cross sectional area of 1 m^2 and are separated by a distance of 0.1 m . What is the value of the electric field near the center of the capacitor?

The surface charge density is given by $\sigma = \frac{Q}{A} = 1 \frac{\mu\text{C}}{\text{m}^2}$.

Outside the capacitor note that the electric field is zero.

Choosing a cylindrical Gaussian surface of area A' we have that between the plates, the electric flux through the ends of the cylinder is given by:

$$EA' = \frac{\sigma A'}{\epsilon_0} \Rightarrow E = \frac{\sigma}{\epsilon_0} = \frac{1 \times 10^{-6} \text{ C}}{8.85 \times 10^{-12} \text{ m}^2} = 1.13 \times 10^5 \frac{\text{N}}{\text{C}}.$$