

Supplementary problems for electrostatics

2016

1. Three identical charges each with $q=1\mu\text{C}$ are arranged at the vertices of an equilateral triangle with a side of length 1m . Find the electric field at the center of the triangle.
2. A square has charges at the following locations and values:
 $(1:1\mu,-1,-1),(2:2\mu,1,-1),(3:3\mu,1,1),(4:4\mu,-1,1)$.
Find the value of the vector electric field at the origin.
3. Three charges are arranged as follows:
 $(1:1\mu,-1,0),(2:-1\mu,1,0),(3:1\mu,0,1)$.
Find the vector electric field at the origin.
4. A sphere of radius a has a charge Q spread uniformly over its volume and at the center of the sphere is a charge $-Q$. Find the vector electric field inside and outside the sphere.
5. A parallel plate capacitor has a total charge $Q=1\mu\text{C}$ on one plate and $Q=-1\mu\text{C}$ on the other plate. The plates have a cross sectional area of 1 m^2 and are separated by a distance of 0.1 m . What is the value of the electric field near the center of the capacitor?

1. Three identical charges each with $q=1\mu\text{C}$ are arranged at the vertices of an equilateral triangle with a side of length 1m . Find the electric field at the center of the triangle.

The hard part here is to locate coordinates of the points. Let the triangle have a side of length L . Locate one point at the origin. Locate the second point at $(L,0)$. We need to find the coordinates of the center, and the 3rd point. The x coordinate of the 3rd point is $x=L/2$. The y coordinate is then given by:

$$L^2 = \left(\frac{L}{2}\right)^2 + y^2 \Rightarrow y^2 = L^2 - \frac{L^2}{4} = 3\frac{L^2}{4} \Rightarrow y = \frac{\sqrt{3}}{2}L.$$

The coordinates of the center of the triangle are then:

$$x = \left(\frac{L}{2}, \frac{\sqrt{3}}{4}L\right).$$

In our present case, $L=1$ m. We thus have the coordinates for this particular orientation:

$$1:(1\mu; 0,0), 2:(1\mu, 1,0), 3:(1\mu, 0.5, \frac{\sqrt{3}}{2}), p:(0.5, \frac{\sqrt{3}}{4})$$

Calculate the following vectors: $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_p, \vec{r}_{1p}, \vec{r}_{2p}, \vec{r}_{3p}$. These vectors are:

$$\vec{r}_p = 0.5\hat{x} + \frac{\sqrt{3}}{4}\hat{y}$$

$$\vec{r}_1 = 0\hat{x} + 0\hat{y}; \vec{r}_{1p} = \vec{r}_p - \vec{r}_1 = 0.5\hat{x} + \frac{\sqrt{3}}{4}\hat{y} - (0\hat{x} + 0\hat{y}) = 0.5\hat{x} + \frac{\sqrt{3}}{4}\hat{y}$$

$$\vec{r}_2 = 1\hat{x} + 0\hat{y}; \vec{r}_{2p} = \vec{r}_p - \vec{r}_2 = \left(0.5\hat{x} + \frac{\sqrt{3}}{4}\hat{y}\right) - (1\hat{x} + 0\hat{y}) = -0.5\hat{x} + \frac{\sqrt{3}}{4}\hat{y}$$

$$\vec{r}_3 = 0.5\hat{x} + \frac{\sqrt{3}}{2}\hat{y}; \vec{r}_{3p} = \vec{r}_p - \vec{r}_3 = \left(0.5\hat{x} + \frac{\sqrt{3}}{4}\hat{y}\right) - \left(0.5\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right) = 0\hat{x} - \frac{\sqrt{3}}{4}\hat{y}$$

Now write E directly.

$$\vec{E}_p = k_\mu \left[\hat{x}(0.5 - 0.5 + 0) + \hat{y}\left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) \right] = k_\mu \left[0\hat{x} - \frac{\sqrt{3}}{4}\hat{y} \right]$$

$$\Rightarrow \vec{E}_p = [0\hat{x} - 3892\hat{y}] \frac{\text{N}}{\text{C}}$$

2. A square has charges at the following locations and values:

$(1:1\mu, -1, -1), (2:2\mu, 1, -1), (3:3\mu, 1, 1), (4:4\mu, -1, 1)$.

Find the value of the vector electric field at the origin.

Notice that if $\vec{r}_p = 0\hat{x} + 0\hat{y}$, then $\vec{r}_{ip} = \vec{r}_p - \vec{r}_i = \vec{0} - \vec{r}_i = -\vec{r}_i$. We can use this to write down the electric field directly:

$$\vec{E}_p = k_\mu \left[\frac{1(1\hat{x} + 1\hat{y})}{[1^2 + 1^2]^{3/2}} + \frac{2(-1\hat{x} + 1\hat{y})}{[2]^{3/2}} + \frac{3(-1\hat{x} - 1\hat{y})}{2\sqrt{2}} + \frac{4(\hat{x} - \hat{y})}{2\sqrt{2}} \right]$$

$$\Rightarrow \vec{E}_p = \frac{k_\mu}{2\sqrt{2}} [\hat{x}(1 - 2 - 3 + 4) + \hat{y}(1 + 2 - 3 - 4)] = \frac{k_\mu}{2\sqrt{2}} (-4\hat{y}) = \frac{-2k_\mu}{\sqrt{2}} \hat{y} = -\sqrt{2}k_\mu \hat{y} = -12713\hat{y} \frac{N}{C}$$

3. Three charges are arranged as follows: $(1:1\mu, -1, 0), (2:-1\mu, 1, 0), (3:1\mu, 0, 1)$.

Find the vector electric field at the origin.

This satisfies the condition $\vec{r}_p = \vec{0}$. We can then write the field directly.

$$\vec{E}_p = k_\mu \left[\frac{1(1\hat{x} + 0\hat{y})}{[1^2 + 0^2]^{3/2}} + \frac{-1(-1\hat{x} + 0\hat{y})}{1} + 1(0\hat{x} - 1\hat{y}) \right] = k_\mu [\hat{x}(1 + 1 + 0) + \hat{y}(0 + 0 - 1)] = k_\mu [2\hat{x} - 1\hat{y}]$$

$$\Rightarrow \vec{E}_p = [17980\hat{x} - 8990\hat{y}] \frac{N}{C}$$

4. A sphere of radius a has a charge Q spread uniformly over its volume and at the center of the sphere is a charge $-Q$. Find the vector electric field inside and outside the sphere.

The (uniform) charge density is given by:

$$\rho = \frac{Q}{\text{Volume}} = \frac{Q}{\frac{4}{3}\pi a^3} = \frac{3Q}{4\pi a^3}$$

Outside the sphere: choose a Gaussian surface of radius r , centered on the sphere of charge. On this Gaussian Surface, $\vec{A} = A\hat{r}$: $|\vec{E}| = \text{constant}$: $\vec{E} = |\vec{E}|\hat{r}$.

$$\text{On the Gaussian surface, } \Phi_E = \sum_{\Delta A_i} \vec{E}_i \cdot \vec{\Delta A}_i = \sum_{\Delta A_i} E \Delta A_i = E \sum_{\Delta A_i} \Delta A_i = E(4\pi r^2)$$

$$\text{Then, from Gauss's law we have } \Phi_E = \frac{Q_{\text{enc}}}{\epsilon_0} \Rightarrow E(4\pi r^2) = \frac{Q}{\epsilon_0} \Rightarrow E_{\text{out}} = \frac{Q}{4\pi\epsilon_0 r^2} \Rightarrow \vec{E}_{\text{out}} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}.$$

Inside the sphere: choose a Gaussian surface of radius r , centered on the sphere of charge. On this Gaussian Surface, $\vec{A} = A\hat{r}$: $|\vec{E}| = \text{constant}$: $\vec{E} = |\vec{E}|\hat{r}$.

Inside the sphere, you are not enclosing all the charge; instead you only enclose charge up to a radius r . Since the charge is uniform, we can calculate the enclosed charge as:

$$Q_{\text{enc}} = \rho \times \text{Volume} = \rho \left(\frac{4}{3}\pi r^3 \right) = \frac{3Q}{4\pi a^3} \left(\frac{4}{3}\pi r^3 \right) = Q \left(\frac{r}{a} \right)^3.$$

Now apply Gauss's law. Again, as above:

$$\Phi_E = \sum_{\Delta A_i} \vec{E}_i \cdot \vec{\Delta A}_i = \sum_{\Delta A_i} E \Delta A_i = E \sum_{\Delta A_i} \Delta A_i = E(4\pi r^2)$$

$$\text{So: } E_{\text{in}}(4\pi r^2) = \frac{Q}{\epsilon_0} \left(\frac{r}{a} \right)^3 \Rightarrow E_{\text{in}} = \frac{Q}{4\pi\epsilon_0} \frac{r}{a^3} \Rightarrow \vec{E}_{\text{in}} = \frac{Q}{4\pi\epsilon_0} \frac{r}{a^3} \hat{r}$$

Notice at the surface, the inside solution is equal to the outside solution.

5. A parallel plate capacitor has a total charge $Q=1\mu\text{C}$ on one plate and $-Q=-1\mu\text{C}$ on the other plate. The plates have a cross sectional area of 1 m^2 and are separated by a distance of 0.1 m . What is the value of the electric field near the center of the capacitor?

The surface charge density is given by $\sigma = \frac{Q}{A} = 1 \frac{\mu\text{C}}{\text{m}^2}$.

Outside the capacitor note that the electric field is zero.

Choosing a cylindrical Gaussian surface of area A' we have that between the plates, the electric flux through the ends of the cylinder is given by:

$$EA' = \frac{\sigma A'}{\epsilon_0} \Rightarrow E = \frac{\sigma}{\epsilon_0} = \frac{1 \times 10^{-6} \text{C}}{8.85 \times 10^{-12} \text{m}^2} = 1.13 \times 10^5 \frac{\text{N}}{\text{C}}.$$