1-D Dirac Delta Function

Consider this:

$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2}$$

In spherical coordinates, with spherical symmetry, this is:

$$\vec{\nabla}=\frac{1}{r}\frac{\partial}{\partial r}\left[r^2\right]\hat{r} \quad \text{so,} \quad \vec{\nabla}\cdot\frac{\hat{r}}{r^2}=\frac{1}{r}\frac{\partial}{\partial r}\left[r^2\left(\frac{1}{r^2}\right)\right]=0$$

According to Gauss's theorem, though:

$$\oiiint_{\text{volume}}\vec{\nabla}\cdot\vec{v}=\oiint_{\text{surface}}\vec{v}\cdot d\vec{A}$$

in spherical coordinates, we then have:

$$\oiiint_{\text{volume}}\vec{\nabla}\cdot\frac{\hat{r}}{r^2}=\oiint_{\text{surface}}\frac{\hat{r}}{r^2}\cdot r^2 dr^2 dr \sin\theta\, dr\, d\theta\, d\phi\, \hat{r}=\int_{\theta=0}^{\theta=\pi}\sin\theta\, d\theta\int_{\phi=0}^{\phi=2\pi}d\phi=4\pi$$

The two results are not in agreement. The only possible conclusion is we took the divergence wrong. In fact, the function diverges at the origin and this is one definition of the Dirac delta function defined by:

$$\delta(x)=\begin{Bmatrix}0 & \text{if } x\neq 0\\ \infty & \text{if } x=0\end{Bmatrix} \quad \text{such that} \quad \int_{-\infty}^{+\infty}\delta(x)dx=1$$

A couple of immediate properties that happen are these:

$$f(x)\delta(x)=f(0)\delta(x) \quad \text{and}$$

$$\int_{-\infty}^{+\infty}f(x)\delta(x)dx=\int_{-\infty}^{+\infty}f(0)\delta(x)dx=f(0)\int_{-\infty}^{+\infty}\delta(x)dx=f(0)$$

We can also shift the delta function:

$$\delta(x-a)=\begin{Bmatrix}0 & \text{if } x\neq a\\ \infty & \text{if } x=a\end{Bmatrix} \quad \text{such that} \quad \int_{-\infty}^{+\infty}\delta(x-a)dx=1$$

$$f(x)\delta(x-a)=f(a)\delta(x-a) \quad \text{and}$$

$$\int_{-\infty}^{+\infty}f(x)\delta(x-a)dx=\int_{-\infty}^{+\infty}f(a)\delta(x-a)dx=f(a)\int_{-\infty}^{+\infty}\delta(x-a)dx=f(a)$$

Let $D_1(x)$ and $D_2(x)$ be two functions simply involving delta functions. Then

$$\int_{-\infty}^{+\infty}f(x)D_1(x)dx=\int_{-\infty}^{+\infty}f(x)D_2(x)dx\Rightarrow D_1(x)=D_2(x)$$

for what your author calls “ordinary” functions f(x).

Example 1.15: show that

for $k \neq 0$, $\delta(kx) = \frac{1}{|k|}\delta(x)$ and in particular $\delta(-x) = \delta(x)$.

consider:

$$\int_{x=-\infty}^{x=+\infty} f(x)\delta(kx)dx$$

Now change variables: let y=kx.

If k>0, then, $\int_{y=-\infty}^{y=+\infty} f\left(\frac{y}{k}\right)\delta(y)\frac{dy}{k} = \frac{1}{k}\int_{y=-\infty}^{y=+\infty} f\left(\frac{y}{k}\right)\delta(y)dy = \frac{1}{k}f(0)$

if k<0, then

$$\int_{y=+\infty}^{y=-\infty} f\left(\frac{y}{k}\right)\delta(y)\frac{dy}{k} = \frac{1}{-k}\int_{y=-\infty}^{y=+\infty} f\left(\frac{y}{k}\right)\delta(y)dy = \frac{1}{|k|}f(0)$$

so since the two integrals are the same, we thus require that the two functions are equal.

Problem 1.45

(a) Show that $x\frac{d\delta(x)}{dx}=-\delta(x)$.

Using your author's hint, we have using integration by parts:

$$\int_{x=-\infty}^{x=+\infty} x\frac{d\delta(x)}{dx}=-\int_{x=-\infty}^{x=+\infty}\delta(x)\left(\frac{dx}{dx}\right)dx+x\frac{d\delta(x)}{dx}\Bigg|_{-\infty}^{+\infty}=-\int_{x=-\infty}^{x=+\infty}\delta(x)dx\Rightarrow x\frac{d\delta(x)}{dx}=-\delta(x) \quad .$$

(b) let θ(x) be the Heviside step function defined by:

$$\theta(x)\equiv\begin{Bmatrix}1 & \text{if } x>0\\ 0 & \text{if } x\le 0\end{Bmatrix} \text{ . Show that } \frac{d\theta}{dx}=\delta(x).$$

Consider f(x) which is bounded so that outside of an arbitrary finite interval, it vanishes. Use integration by parts to obtain:

$$\int_{x=-\infty}^{x=+\infty}\frac{d(f(x)\theta(x))}{dx}dx=\int_{x=-\infty}^{x=+\infty} f\frac{d\theta(x)}{dx}dx+\int_{x=-\infty}^{x=+\infty}\theta(x)\frac{df}{dx}dx$$

$$\Rightarrow\left[f(x)\theta(x)\right]_{-\infty}^{+\infty}=\int_{x=-\infty}^{x=+\infty} f\frac{d\theta(x)}{dx}dx+\int_{x=-\infty}^{x=+\infty}\theta(x)\frac{df}{dx}dx$$

$$\left[f(x)\theta(x)\right]_{-\infty}^{+\infty}=0$$

$$\Rightarrow\int_{x=-\infty}^{x=+\infty} f\frac{d\theta(x)}{dx}dx=-\int_{x=-\infty}^{x=+\infty}\theta(x)\frac{df}{dx}dx$$

$$\int_{x=-\infty}^{x=+\infty}\theta(x)\frac{df}{dx}dx=\int_{x=0}^{x=+\infty}\frac{df}{dx}dx=f(+\infty)-f(0)=-f(0)$$

$$\Rightarrow\int_{x=-\infty}^{x=+\infty} f\frac{d\theta(x)}{dx}dx=f(0)$$

But, this is also true:

$$f(0)=\int_{x=-\infty}^{x=+\infty} f(x)\delta(x)dx$$

we thus have by the conditions earlier:

$$\frac{d\theta(x)}{dx}=\delta(x)$$