

Calculate the potential due to a charged thick disk of radius a along the symmetry axis.

$$V(\vec{r}_p) = \int_{\substack{\text{all} \\ q}} \frac{k dq_i}{|\vec{r}_p|}$$

coordinates:

$$\vec{r}_i = x_i \hat{x} + y_i \hat{y} : \vec{r}_p = z_p \hat{z} : \vec{r}_{ip} = -x_i \hat{x} - y_i \hat{y} + z_p \hat{z} : |\vec{r}_{ip}| = \sqrt{x_i^2 + y_i^2 + z_p^2} = \sqrt{s_i^2 + z_p^2}$$

dq:

$$dq_i = \rho s ds d\phi dz_i$$

Then:

$$V(\vec{r}_p) = \int_{z=0}^{z=h} \int_{s=0}^{s=a} \int_{\phi=0}^{\phi=2\pi} \frac{k s ds d\phi dz_i}{\sqrt{s^2 + (z_p - z_i)^2}} = 2\pi k \rho \int_{z_i=0}^{z_i=h} \left\{ \sqrt{a^2 + (z_p - z_i)^2} - \sqrt{(z_p - z_i)^2} \right\} dz_i$$

Consider only (here) the case for $z_p > z_i$

$$V(\vec{r}_p) = 2\pi k \rho \int_{z_i=0}^{z_i=h} \left\{ \sqrt{a^2 + (z_p - z_i)^2} - (z_p - z_i) \right\} dz_i$$

It's probably easiest to see this by changing variables:

$$\text{let } u \equiv z_p - z_i : du = -dz_i : z_i = 0 \Rightarrow u = z_p : z_i = h \Rightarrow u = z_p - h$$

$$\begin{aligned} V(\vec{r}_p) &= 2\pi k \rho \left[- \int_{u=z_p}^{u=z_p-h} \sqrt{a^2 + u^2} du + \int_{u=z_p}^{u=z_p-h} u du \right] = \\ &= \left\{ 2\pi k \rho \left[-\frac{1}{2} \left(\ln a^2 \left[u + \sqrt{a^2 + u^2} \right] + u \sqrt{a^2 + u^2} \right) \right] + 2\pi k \rho \frac{u^2}{2} \right\} \Bigg|_{u=z_p}^{u=z_p-h} \end{aligned}$$

The unbeautiful result is:

$$V(\vec{r}_p) = \pi k \rho \left\{ \left[\ln \frac{z_p + \sqrt{a^2 + z_p^2}}{z_p - h + \sqrt{a^2 + (z_p - h)^2}} + z_p \sqrt{a^2 + z_p^2} - (z_p - h) \sqrt{a^2 + (z_p - h)^2} \right] + [-2z_p h + h^2] \right\}$$