To show: For the angular portion of Laplace's equation:

$$\frac{1}{\Theta}\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right)=-m(m+1)$$

solutions are given by the Legendre polynomials defined by:

$$P_m(x)=\frac{1}{2^m m!}\left(\frac{d}{dx}\right)^m\left(x^2-1\right)^m$$

where $x \equiv \cos(\theta)$

Let's rewrite Laplace's equation above:

$$d(\cos(\theta))=-\sin\theta\, d\theta \Rightarrow d\theta=-\frac{dx}{\sin(\theta)} \Rightarrow \frac{d}{d\theta}=-\sin(\theta)\frac{d}{dx}$$

so we then have:

$$\sin(\theta)\frac{d\Theta}{d\theta}=\sin(\theta)\left[-\sin(\theta)\frac{d\Theta}{dx}\right]=-\sin^2\theta\frac{d\Theta}{dx}$$

Then we have:

$$\frac{1}{\Theta}\frac{1}{\sin(\theta)}\left\{-\sin(\theta)\frac{d}{dx}\right\}\left(-\sin^2\theta\frac{d\Theta}{dx}\right)=-m(m+1)$$

$$\frac{1}{\Theta}\frac{d}{dx}\left(\left[1-\cos^2\theta\right]\frac{d\Theta}{dx}\right)=-m(m+1)$$

$$\frac{1}{\Theta}\frac{d}{dx}\left(\left[1-x^2\right]\frac{d\Theta}{dx}\right)=-m(m+1)$$

or:

$$\frac{d}{dx}\left(\left[1-x^2\right]\frac{d\Theta}{dx}\right)=-m(m+1)\Theta$$

is the differential equation we wish to solve and in particular, we what to show it has as its solution the Legendre polynomials:

$$P_m(x)=\frac{1}{2^m m!}\left(\frac{d}{dx}\right)^m\left(x^2-1\right)^m$$

One additional helpful thing that can be shown is the derivative:
(reference Mathworld here)

$$\left(1-x^2\right)\frac{dP_m(x)}{dx}=-mxP_m(x)+mP_{m-1}(x)=(m+1)xP_m(x)-(m+1)P_{m+1}(x)$$

Then for $P_{m-1}(x)$ we have:

$$\left(1-x^2\right)\frac{dP_{m-1}(x)}{dx}=-(m-1)xP_{m-1}(x)+(m-1)P_{m-2}(x)=(m)xP_{m-1}(x)-(m)P_m(x)$$

To show then the required showing, I prefer to use these rules for the derivatives. What we then require is this which is from the first rule which was:

$$\left(1-x^2\right)\frac{dP_m(x)}{dx} = -mxP_m(x) + mP_{m-1}(x) = (m+1)xP_m(x) - (m+1)P_{m+1}(x)$$

This gives:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_m(x)}{dx}\right] = \frac{d}{dx}\left[-mxP_m(x) + mP_{m-1}(x)\right] = -mP_m(x) - mx\frac{dP_m(x)}{dx} + m\frac{dP_{m-1}(x)}{dx}$$

We can now evaluate this derivative using the rule above:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_m(x)}{dx}\right] =$$

$$-mP_m(x)$$

$$-mx\frac{dP_m(x)}{dx} = -m\frac{x}{1-x^2}\left[\left(1-x^2\right)\frac{dP_m(x)}{dx}\right] = -m\frac{x}{1-x^2}\left\{-mxP_m(x) + mP_{m-1}(x)\right\}$$

$$m\frac{dP_{m-1}(x)}{dx} : m\frac{1}{\left(1-x^2\right)}\left[\left(1-x^2\right)\frac{dP_{m-1}(x)}{dx}\right] = m\frac{1}{\left(1-x^2\right)}\left\{(m)xP_{m-1}(x) - (m)P_m(x)\right\}$$

Now we add everything up to evaluate the result:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_m(x)}{dx}\right] = P_m(x)\left\{-m + m^2\frac{x^2}{1-x^2} - m^2\frac{x^2}{1-x^2}\right\} + P_{m-1}(x)\left\{-m^2\frac{x}{1-x^2} + m^2\frac{x}{1-x^2}\right\}$$

$$= \left\{-m + m^2\frac{x^2}{1-x^2} - m^2\frac{x^2}{1-x^2}\right\}P_m(x) = \left\{-m + m^2\frac{x^2-1}{1-x^2}\right\}P_m(x) = -\left[m + m^2\right]P_m(x)$$

$$\Rightarrow \frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_m(x)}{dx}\right] = -m(m+1)P_m(x)$$

Which is thus the conclusion that was desired. There is also a neat way to show that the Legendre Polynomials are defined by the generating function:

$$g(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{m=0}^{\infty} P_m(x)t^m$$

For additional details I'll refer you to this [link]()