

Divergence and Curl of Electrostatic Fields

I'm diverging from your textbook presentation here to use some of what we've developed.

We know that the electric field is given by:

$$\vec{E}_p = \int_{\text{all charges}} k \frac{dq}{(\vec{r}_p - \vec{r}_i)^2} \hat{r}_{ip}$$

or in the discrete case, we have:

$$\vec{E} = \sum_{\text{all charges}} k \frac{q_i}{(\vec{r}_p - \vec{r}_i)^2} \hat{r}_{ip}$$

Let's assume that we can write the charge distribution for the continuum case as a volume charge distribution. Then:

$$\vec{E}_p = \int_{\text{all space}} k \frac{\rho(\vec{r}) d\tau}{(\vec{r}_p - \vec{r}_i)^2} \hat{r}_{ip}$$

where we are permitted to integrate over all space if the volume distribution correctly specifies that the distribution is zero when we are in regions of space where the charges are not.

Probably one simple example of a charge distribution which you have eagerly awaited is that of a point charge:

$$\rho(\vec{r}_i) = \frac{q}{4\pi} \delta(\vec{r}_i)$$

This would refer to a point charge of strength q located *at the origin*. Let's show quickly that this gives the correct electric field:

$$\vec{E}_p = \int_{\text{all charges}} k \frac{\rho(\vec{r}_i) d\tau_i}{(\vec{r}_p - \vec{r}_i)^2} \hat{r}_{ip} = \int_{\text{all space}} k \frac{\frac{q}{4\pi} \delta(\vec{r}_i) d\tau_i}{(\vec{r}_p - \vec{r}_i)^2} \hat{r}_{ip} = k \frac{q}{|\vec{r}_p|^2} \hat{r}_{ip} = \frac{kq}{|\vec{r}|^2} \hat{r}$$

You can also see that if we have a series of n point charges, we could express the charge distribution as:

$$\rho(\vec{r}_i) = \frac{1}{4\pi} \sum_{i=1}^n q_i \delta(\vec{r}_i)$$

With this, the superposition of the electric field is clearly seen:

$$\vec{E}_p = \int_{\text{all charges}} k \frac{\rho(\vec{r}_i) d\tau_i}{(\vec{r}_p - \vec{r}_i)^2} \hat{r}_{ip} = \int_{\text{all space}} k \frac{\frac{1}{4\pi} \sum_{i=1}^n q_i \delta(\vec{r}_i) d\tau_i}{|\vec{r}_p - \vec{r}_i|^2} \hat{r}_{ip} = k \sum_{i=1}^n \frac{q_i}{|\vec{r}_p - \vec{r}_i|^2} \hat{r}_{ip}$$

The question of switching the orders of integration and summation hinges upon the question of completeness, for an infinite series.

For a finite series, such as this, there is no problem.

In order to apply this to both sides of the equation, we're going to need to write the charge distribution as a divergence. Thus we can most certainly write:

$$\rho(\vec{r}_i) = \frac{1}{4\pi} \sum_{i=1}^n q_i \delta^3(\vec{r}_i)$$

But we know that the delta function can be written as a divergence:
(see Equation 1.100):

$$\vec{\nabla}_{\vec{r}_p} \cdot \left(\frac{\hat{r}_{ip}}{r_{ip}^2} \right) = 4\pi\delta^3(\vec{r}_{ip})$$

The subscript on the divergence operator indicates (in accord with Equation 1.100) the differentiation here is with respect to the space (point) coordinates $\vec{r}_p$ while the charge coordinates $\vec{r}_i$ are held fixed. This subtle but important point needs to be rigorously remembered.

So:

$$\int_{\text{all charges}} \rho(\vec{r}_i) d\tau_i = \int_{\text{all space}} \vec{\nabla}_{\vec{r}_p} \cdot \sum_{i=1}^n \frac{q_i}{4\pi} \left(\frac{\hat{r}_{ip}}{r_{ip}^2} \right) d\tau$$

Let's put it all together now:

We can write:

$$\rho(\vec{r}_i) = \vec{\nabla}_{\vec{r}_p} \cdot \left[\sum_{i=1}^n \frac{q_i}{4\pi} \left(\frac{\hat{r}_{ip}}{r_{ip}^2} \right) \right]$$

But we know that the electric field is given exactly by:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n q_i \left(\frac{\hat{r}_{ip}}{r_{ip}^2} \right) \Rightarrow \sum_{i=1}^n \frac{q_i}{4\pi} \left(\frac{\hat{r}_{ip}}{r_{ip}^2} \right) = \epsilon_0 \vec{E}$$

We thus have:

$$\vec{\nabla}_{\vec{r}_p} \cdot (\epsilon_0 \vec{E}) = \rho(\vec{r}_i)$$

This is the differential form of Gauss's law.

We can easily convert it into an integral form by performing an integral over all (point) space:

$$\oint_{\vec{r}_p} \vec{\nabla}_{\vec{r}_p} \cdot (\epsilon_0 \vec{E}) d\tau_{\vec{r}_p} = \oint_{\vec{r}_p} \rho(\vec{r}_i) d\tau_{\vec{r}_p}$$

I am probably being overly cautious here to specify the integration is over the space and not the charge coordinates here. Since you're integrating over all space, however, the charge coordinates enclosed in this are necessarily a subset of this space.

Looking at the useful sheet, we can see that the left hand side of this integral can be converted into a surface integral:

$$\int_{\text{volume}} (\vec{\nabla} \cdot \vec{V}) d\tau = \oint_{\text{surface}} \vec{V} \cdot d\vec{A}$$

Thus, we can write:

$$\oint_{\text{surface}} \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \oint_{\text{volume}} \rho(\vec{r}_i) d\tau_{\vec{r}_p} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Ok, it's pretty straight-forward to see then that the integral form of Gauss's law becomes:

$$\Phi_E = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Where the electric flux is defined by:

$$\Phi_E \equiv \oint_S \vec{E} \cdot d\vec{A}$$

We have covered Gauss's law in Physics 220 and Physics 250. I will refer you to the notes on those courses in order to see examples for the application of Gauss's Law in simple situations.

In the Electrostatic situation, it's not too hard to show that the curl of $\vec{E}$ is zero.

Let's see how:

Consider the electric field arising from a point charge:

$$\vec{E} = \sum_{i=1}^n \frac{q_i}{4\pi\epsilon_0} \left(\frac{\hat{r}_{ip}}{r_{ip}^2} \right)$$

For a single point charge located at the origin, we have:

$$\vec{E}_p = \frac{q}{4\pi\epsilon_0 r_p^2} \hat{r}$$

Let's look at the integral of E on a path:

in the spherical coordinate system, the useful sheet says:

$$d\vec{l} = dr\hat{r} + r d\theta\hat{\theta} + r \sin(\theta) d\phi\hat{\phi}$$

thus, we can write:

$$\vec{E} \cdot d\vec{l} = \frac{q}{4\pi\epsilon_0 r^2} dr$$

So, from a point a to a point b we have:

$$\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b \frac{q}{4\pi\epsilon_0 r^2} dr = -\frac{q}{4\pi\epsilon_0 r} \Big|_a^b = \frac{q}{4\pi\epsilon_0 a} - \frac{q}{4\pi\epsilon_0 b}$$

Now for an arbitrary closed path, it is abundantly clear that:

$$\oint_P \vec{E} \cdot d\vec{l} = 0$$

Warning!!! this only applies to electrostatic fields!

Now, look at Stokes' theorem from the useful sheet:

$$\int_{\text{surface}} (\vec{\nabla} \times \vec{V}) \cdot d\vec{A} = \oint_{\text{path}} \vec{V} \cdot d\vec{l}$$

Note: The surface here is singly connected (not a Klein Bottle type thing) and is bounded by the path. However, other than that, there is no requirement that it have any particular shape: indeed if your path were circular, the area integrated over could easily be a hemisphere or a disk or other shapes. I think, however, it is probably best to avoid a surface that extends to infinity and back bounded by a curve.

Let's convert this via this theorem:

$$\oint_P \vec{E} \cdot d\vec{l} = \oint_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{A}$$

Thus, you can clearly see the implication:

$$\vec{\nabla} \times \vec{E} = \vec{0}$$

now, look at the implication of this (eq. 1.103)

$$\vec{\nabla} \times \vec{E} = \vec{0} \Rightarrow \vec{E} = -\vec{\nabla} V$$

Now, that little path integral above will fail in an anticipated case ... namely when we look at the emf produced by a time rate of change of magnetic flux. That means, things will get more complicated for time-dependant fields.

(This is going to involve a more general vector field).

look at eq. 1.105:

$$\vec{F} = -\vec{\nabla} V + \vec{\nabla} \times \vec{A}$$

Also, since this is true for one charge, it's true for them all via superposition.

Which brings us up to the calculation of this electric potential.

Since we now know:

$$\vec{E} = -\vec{\nabla} V$$

and looking at the gradient theorem from the useful sheet:

$$\int_{\substack{a \\ \text{path}}}^b (\vec{\nabla} T) \cdot d\vec{l} = T(b) - T(a)$$

we can say:

$$\int_a^b \vec{E} \cdot d\vec{l} = V(a) - V(b)$$

We can also choose any path so let's pick this one:

$$-\int_{\emptyset}^r \vec{E} \cdot d\vec{l} = V(r)$$

where the funny symbol refers to some arbitrarily chosen point.