Problem 1.58

$$\int\limits_{\text{volume}} \left(\vec{\nabla} \bullet \vec{V}\right) d\tau = \oint\limits_{\text{surface}} \vec{V} \bullet d\vec{A}$$

$$\vec{v} = r^2 \sin\theta \hat{r} + 4r^2 \cos\theta \hat{\theta} + r^2 \tan\theta \hat{\varphi}$$

$$\vec{\nabla} \bullet \vec{V} = \tfrac{1}{r^2} \tfrac{\partial}{\partial r}\left(r^2 V_r\right) + \tfrac{1}{r\sin(\theta)} \tfrac{\partial}{\partial \theta}\left(\sin(\theta) V_\theta\right) + \tfrac{1}{r\sin(\theta)} \tfrac{\partial V_\varphi}{\partial \varphi}$$

So:

$$\vec{\nabla} \bullet \vec{v} = \tfrac{1}{r^2} 4r^3 \sin\theta + \tfrac{4r^2}{r\sin\theta}\left[-\sin^2\theta + \cos^2\theta\right] =$$

$$= 4r\sin\theta + 4r\left[-\sin\theta + \tfrac{\cos^2\theta}{\sin\theta}\right] = 4r\cos\theta\cot\theta$$

$$\oint\limits_{v}\left[\vec{\nabla} \bullet \vec{v}\right] d\tau = \int\limits_{r=0}^{R} \int\limits_{\theta=0}^{\frac{\pi}{6}} \int\limits_{\varphi=0}^{2\pi} 4r\cos\theta\cot\theta \bullet r^2 \sin(\theta) dr d\theta d\varphi$$

$$\oint\limits_{v}\left[\vec{\nabla} \bullet \vec{v}\right] d\tau = \int\limits_{r=0}^{R} \int\limits_{\theta=0}^{\frac{\pi}{6}} \int\limits_{\varphi=0}^{2\pi} 4r^3 \cos^2\theta dr d\theta d\varphi =$$

$$2\pi R^4 \int\limits_{\theta=0}^{\theta=\frac{\pi}{6}} \cos^2\theta d\theta = 2\pi R^4 \tfrac{1}{2}\left[\theta + \cos(\theta)\sin\theta\right]\Big|_{\theta=0}^{\theta=\frac{\pi}{6}}$$

$$= \pi R^4\left[\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2}\tfrac{1}{2}\right] = \pi R^4\left[\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{4}\right] = \tfrac{\pi R^4}{12}\left[2\pi + 3\sqrt{3}\right]$$

It is more difficult to verify the divergence theorem here since it is hard to define the normal to the areas of the cone.