1.56

$\vec{v} = \left(r\cos^2\theta\right)\hat{r} - \left(r\cos\theta\sin\theta\right)\hat{\theta} + 3r\hat{\varphi}$

Curve 1: (0,0,0)->(1,0,0): $d\vec{l} = dx\hat{x}$

$\hat{r} = \sin(\theta)\cos(\varphi)\,\hat{x} + \sin(\theta)\sin(\varphi)\,\hat{y} + \cos(\theta)\,\hat{z}$

$\hat{\theta} = \cos(\theta)\cos(\varphi)\,\hat{x} + \cos(\theta)\sin(\varphi)\,\hat{y} - \sin(\theta)\,\hat{z}$

$\hat{\varphi} = -\sin(\varphi)\,\hat{x} + \cos(\varphi)\,\hat{y}$

$\hat{r} \bullet \hat{x} = \sin(\theta)\cos(\varphi)$

$\hat{\theta} \bullet \hat{x} = \cos(\theta)\cos(\varphi)$

$\hat{\varphi} \bullet \hat{x} = -\sin(\varphi)\,\hat{x}$

$\vec{v} \bullet d\vec{l} = \left(r\cos^2\theta\right)\sin(\theta)\cos(\varphi) - \left(r\cos\theta\sin\theta\right)\cos(\theta)\cos(\varphi) - 3r\sin(\varphi)$

along x axis: $\theta = \frac{\pi}{2} : \varphi = 0 \Rightarrow \vec{v} \bullet d\vec{l} = 0 \Rightarrow \int_1 \vec{v} \bullet d\vec{l} = 0$

Along curve:

$d\vec{l} = dx\hat{x} + dy\hat{y}$ but $\theta = \frac{\pi}{2}$ so the only component of v that survives is the phi component.

Thus:

$\vec{v} \bullet d\vec{l} = 3r\hat{\varphi} \bullet \left[dx\hat{x} + dy\hat{y}\right] = 3r\left[-\sin\varphi dx + \cos\varphi dy\right]$

The radius of this path is 1. Convert to polar coordinates:

$x = s\cos\varphi \Rightarrow dx = -s\sin\varphi d\varphi \Rightarrow dx = -\sin\varphi d\varphi$

$y = s\sin\varphi \Rightarrow dy = s\cos\varphi d\varphi \Rightarrow dy = \cos\varphi d\varphi$

$\Rightarrow \vec{v} \bullet d\vec{l} = 3r\left[\sin^2\varphi + \cos^2\varphi\right] = 3r$

$\Rightarrow \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \vec{v} \bullet d\vec{l} = 3\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$

On the third path: y=1, x=0 and z varies between 0 and 2.

Thus: $d\vec{l} = dz\hat{z}$

Also in this plane, φ=90 degrees so

$\hat{r} \bullet \hat{z} = \cos(\theta)$

$\hat{\theta} \bullet \hat{z} = -\sin(\theta)$

$\hat{\varphi} \bullet \hat{z} = 0$

$\vec{v} \bullet d\vec{l} = \left(r\cos^2\theta\right)\cos\theta + \left(r\cos\theta\sin\theta\right)\sin\theta$

$= r\cos^2\theta\cos\theta + r\cos\theta\left[1 - \cos^2\theta\right]$

$= r\cos\theta = z$

$\vec{v} = \left(r\cos^2\theta\right)\hat{r} - \left(r\cos\theta\sin\theta\right)\hat{\theta} + 3r\hat{\varphi}$

$$\cos\theta = \frac{z}{\sqrt{y^2+z^2}} : r = \sqrt{y^2+z^2} : y = 1 \Rightarrow$$

Now $\Rightarrow \cos\theta = \frac{z}{\sqrt{1+z^2}} : r = \sqrt{1+z^2} \Rightarrow r\cos\theta = z$

$$\int_{z=0}^{z=2} \vec{v} \bullet d\vec{l} = \int_{z=0}^{z=2} z dz = \frac{z^2}{2}\Big|_0^2 = \frac{4}{2} = 2$$

$$\vec{v} = \left(r\cos^2\theta\right)\hat{r} - \left(r\cos\theta\sin\theta\right)\hat{\theta} + 3r\hat{\varphi}$$

Along the last segment:

$$d\vec{l} = dy\hat{y} + dz\hat{z}$$

$$\vec{v} = \left(r\cos^2\theta\right)\hat{r} - \left(r\cos\theta\sin\theta\right)\hat{\theta} + 3r\hat{\varphi}$$

$$\hat{r} = \sin(\theta)\cos(\varphi)\hat{x} + \sin(\theta)\sin(\varphi)\hat{y} + \cos(\theta)\hat{z}$$

$$\hat{\theta} = \cos(\theta)\cos(\varphi)\hat{x} + \cos(\theta)\sin(\varphi)\hat{y} - \sin(\theta)\hat{z}$$

$$\hat{\varphi} = -\sin(\varphi)\hat{x} + \cos(\varphi)\hat{y}$$

$$\varphi = \frac{\pi}{2} \Rightarrow$$

$$\hat{r} = 0\hat{x} + \sin(\theta)\hat{y} + \cos(\theta)\hat{z}$$

$$\hat{\theta} = 0\hat{x} + \cos(\theta)\hat{y} - \sin(\theta)\hat{z}$$

$$\hat{\varphi} = -\hat{x} + 0\hat{y}$$

$$\hat{r} \bullet \hat{y} = \sin\theta : \hat{r} \bullet \hat{z} = \cos\theta \Rightarrow \hat{r} \bullet d\vec{l} = \sin\theta dy + \cos\theta dz$$

$$\hat{\theta} \bullet \hat{y} = \cos\theta : \hat{\theta} \bullet \hat{z} = -\sin\theta \Rightarrow \hat{\theta} \bullet d\vec{l} = \cos\theta dy - \sin\theta dz$$

$$\hat{\varphi} \bullet \hat{y} = 0 : \hat{\varphi} \bullet \hat{z} = 0$$

$$\vec{v} \bullet d\vec{l} = \left(r\cos^2\theta\right)\left[\sin\theta dy + \cos\theta dz\right] - \left(r\cos\theta\sin\theta\right)\left[\cos\theta dy - \sin\theta dz\right]$$

$$= r\cos^2\theta\sin\theta dy + r\cos^3\theta dz - r\cos^2\theta\sin\theta dy + r\cos\theta\sin^2\theta dz$$

$$= r\cos\theta\left[\cos^2\theta + \sin^2\theta\right]dz = z dz \Rightarrow \int_2^0 z dz = -2$$

So

The complete answer: $I = \frac{3\pi}{2} + 2 - 2 = \frac{3\pi}{2}$

Stokes' Theorem:

$$\int_{\text{surface}} \left(\vec{\nabla} \times \vec{V}\right) \bullet d\vec{A} = \oint_{\text{path}} \vec{V} \bullet d\vec{l}$$

In the plane, θ=π/2

$$\Rightarrow \vec{v} = 0\hat{r} - 0\hat{\theta} + 3r\hat{\varphi}$$

$$\hat{r} = \sin(\theta)\cos(\varphi)\hat{x} + \sin(\theta)\sin(\varphi)\hat{y} + \cos(\theta)\hat{z}$$

But $\hat{\theta} = \cos(\theta)\cos(\varphi)\hat{x} + \cos(\theta)\sin(\varphi)\hat{y} - \sin(\theta)\hat{z}$

$$\hat{\varphi} = -\sin(\varphi)\hat{x} + \cos(\varphi)\hat{y}$$

$$\vec{v} = 3\sqrt{x^2 + y^2}\left[-\tfrac{y}{\sqrt{x^2+y^2}}\,\hat{x} + \tfrac{x}{\sqrt{x^2+y^2}}\,\hat{y}\right] = -3y\hat{x} + 3x\hat{y}$$

so

$$\left[\vec{\nabla}\times\vec{v}\right]\bullet\hat{z} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3y & 3x & 0 \end{vmatrix} = \hat{z}\left[3 - -3\right] = 6\hat{z} \Rightarrow \oint\left[\vec{\nabla}\times\vec{v}\right]\bullet\hat{z}dA = 6\left\{\tfrac{\pi}{4}\right\} = \tfrac{3\pi}{2}$$

In the y-z plane:

$$\vec{v} = \left(r\cos^2\theta\right)\hat{r} - \left(r\cos\theta\sin\theta\right)\hat{\theta} + 3r\hat{\varphi}$$

$$\vec{\nabla}\times\vec{V} = 0\hat{r} + \tfrac{1}{r}\left[0 - 6r\right]\hat{\theta} + \tfrac{1}{r}\left[0\right]\hat{\varphi} = -6\hat{\theta}$$

$$d\vec{A} = dydz\hat{x} \Rightarrow \left[\vec{\nabla}\times\vec{v}\right]\bullet d\vec{A} = -6\hat{\theta}\bullet\hat{x}dydz$$

$$\hat{r} = \sin(\theta)\cos(\varphi)\,\hat{x} + \sin(\theta)\sin(\varphi)\,\hat{y} + \cos(\theta)\,\hat{z}$$

$$\hat{\theta} = \cos(\theta)\cos(\varphi)\,\hat{x} + \cos(\theta)\sin(\varphi)\,\hat{y} - \sin(\theta)\,\hat{z}$$

$$\hat{\varphi} = -\sin(\varphi)\,\hat{x} + \cos(\varphi)\,\hat{y}$$

$\hat{\theta}\bullet\hat{x} = \cos(\theta)\cos(\varphi)$ and when $\varphi = \frac{\pi}{2} \Rightarrow \hat{\theta}\bullet\hat{x} = 0$ so I=0 also.

The answers are thus in agreement.