A simple consequence of everyone measuring the same value for the speed of light r

Revised 2024

Consider two observers, Frank in system O and Mary in system O'. Let Mary move at some velocity $v=\beta c$ relative to Frank along the x-axis. Assume that at t=0, Mary's frame is accelerated so that the average velocity of her frame for the duration of this observation is $v=\beta c$ while Frank's frame is regarded as inertial. Both Frank and Mary are holding lasers. Let Frank only pulse a laser.

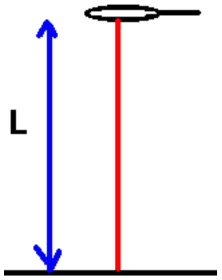
Now we are not going to be worrying about how large the time increments are; make them small.

But remember, Mary has moved the distance x_m that Frank has not moved through.

How can both observers possibly report the same results?

The answer can only be that a time interval of 1s in Mary's frame can not be the same as a time interval of 1s in Frank's frame. According to Frank, Mary's clock must be running slow. But, Mary says that Frank's meter stick is too short. This is the downfall of simultaneity: two events which are simultaneous in one reference frame are not necessarily simultaneous in another reference frame.

Frank has a mirrors floating over his head.



The time it takes a pulse to travel through this distance and back is $t_0=\frac{2L}{c}$. or not. An observer in Frank's frame of reference will measure this time.

Proper time t_0 is the time interval between two events measured by an observer who is at rest relative to the two events and sees the events occur at the same point in space. There is the requirement for this frame to be inertial.

Now let the frame of reference move (with Mary) at an average velocity of $v=\beta c$ along the x-direction (perpendicular to the direction of the laser pulse). We need to determine what Mary observes for Frank's pulse.

The additional distance can only be accounted for if somehow Mary's clock ran slower than Frank's clock. But now, we can calculate this exactly. Then, using the time dilation, we can return to the previous problem to obtain the length contraction.

Frank sees the blue line up and back and the distance that the light traveled is $2ct_f$. In the time (t_m), Mary saw Frank's frame move through a distance $2\beta c t_m$. Mary measured the pulse to travel through the red path so we want to find out what path length Mary measured. These three quantities are related by Pythagorean's theorem:

$$c^2 t_m^2 = \beta^2 c^2 t_m^2 + c^2 t_f^2$$

We want to solve this for t. The result is:

$$c^2\left[1-\beta^2\right]t_m^2 = c^2 t_f^2 \Rightarrow t_m = \frac{1}{\sqrt{1-\beta^2}} t_f \Rightarrow t_m = \gamma t_f$$

In this expanded example, Frank's time increment is set to be the proper time. This means that Mary's time increment is going to be larger. The factor γ is always greater than or equal to 1. This means that for every tick of Mary's clock, Frank's clock ticks more. In other words, Mary's clock runs slow according to Frank. Now if we go the other way around, since Mary's frame is also inertial, Mary sees Frank's clock run slow also.

Now let's try to make a length measurement.
Proper length L_0 is the distance between two points in a frame of reference which is at rest with respect to those two points.

How to make a length measurement.
Let Mary be walking with an average speed given by $v=\beta c$ parallel to a stick and she is carrying a pointer. Frank is in the reference frame of the stick. Mary measures the length of the stick by determining how long it takes for her to walk past the stick, according to a stopwatch in her moving frame of reference. She does this by noticing when her pointer is at one end of the stick (thus starting her stopwatch) and then noticing when her pointer is at the other end of the stick (thus stopping her stopwatch). This distance is given by $L_m = \beta c t_m$. But the time dilates as

$$t_m = \gamma t_f$$

So

$$L_m = \gamma \beta c t_f$$

Frank also measures the length of the stick by noting when Mary's pointer is at each end of the stick. Frank measures the following result: $L_f = \beta c t_f$.
Now we need to determine which of these times is the proper time in order to relate the two measurements. We need to look at how proper time is defined to determine this.

Proper time t_0 is the time interval between two events measured by an observer who is at rest relative to the two events and sees the events occur at the same point in space. This frame is inertial.

Now it is Mary who is looking at the same pointer in her reference frame. Frank has to rotate his head to follow Mary's pointer so clearly Frank’s time can not be the proper time. By process of elimination, it must be Mary’s time which is therefore the proper time. Don’t worry about the fact that Frank sees Mary’s frame moving, Mary sees her frame at rest and she sees the defining events at the same location in space. This then gives us the connection between Frank’s time increment and Mary’s time increment:

$$t_f = t : t_m = t_0 \Rightarrow t_f = \gamma t_m = \gamma t_0$$

We can now relate the two lengths:

$$L_m = \beta c t_0$$
$$L_f = \beta c t_f = \beta c \gamma t_0$$

Now let’s determine which of these lengths is the proper length. Again, looking at the definition we have:

Proper length L_0 is the distance between two points in a frame of reference which is at rest with respect to those two points.

Clearly the distance in Frank’s frame of reference is the proper length because it is his reference frame which is at rest with respect to the stick. This means Frank measures L_0.

So this is then the way the measurements are related:

$$L_m = \beta c t_0$$
$$L_0 = \beta c \gamma t_0$$

We divide the two to obtain:

$$\frac{L_m}{L_0} = \frac{\beta c t_0}{\beta c \gamma t_0} = \frac{1}{\gamma} \Rightarrow L_m = \frac{1}{\gamma} L_0$$

I think we can now lose the “m” subscript. We can simply say then:

$$L = \frac{1}{\gamma} L_0 \quad .$$

Again, if you want to describe the correct length measurement procedure, you need to see (for now) the two measurements happen at the same time.

This contracted length is always shorter than or equal to the proper length.

Remember the time dilation is given by:

$$t = \gamma t_0$$

which is always greater than or equal to the proper time. Again, note that this is describing Mary's time as seen by Frank.

I guess you can remember:

moving clocks run slow and moving meter sticks expand.

There is an important warning: If two events are separated by both time and space (meaning they don't occur at the same location in space), they can not be related simply by multiplying of dividing by the gamma factor.

Here are some simple problems to get started on:

(1) Frank measures a time interval of 5s on his watch during which time Mary's spaceship accelerated and is moving on a return trip at an average speed of 0.9999c. We can imagine that Mary is moving in a circle if you find this easier to visualize. When the two again compare their clocks, what do they read?
Revised 2024

Since Mary's frame is not an inertial frame it is Frank's clock that will be ticking off time more rapidly.

$$t_m = \gamma t_f = \gamma t_0 \ .$$

Even more explicitly: let 1 sec tick on Frank's clock. In that time frame, Mary's clock only ticked 0.014 sec.

The gamma factor here is $\gamma = \frac{1}{\sqrt{1-0.9999^2}} = 70.7$ which is the factor that tells us that a 1 sec tick on Frank's clock corresponds to 1/70.7 as seen on Mary's clock. So 5 seconds on Frank's clock would give only 5x0.014 = 0.07 s on Mary's clock. So when they again compare clocks, Frank reads 5 sec, Mary reads 0.07 s. So now, if 1 second elapsed on Mary's clock, how much time elapsed on Frank's clock?

If both clocks are in inertial reference frames, then two clocks moving relative to each other will both see the other ticking off more slowly and both frames have proper time. However, in order to compare the two clocks, one frame would have to become non-inertial which would violate the assumption of each frame being an inertial frame.

In the earlier notes, emphasis needs to be placed on the idea that the proper time is also in that frame which is inertial.

(2) How fast does a rocket need to go so that the occupant of the ship lives 4 times the normal lifespan of a fixed inertial observer according to the fixed observer?

This is asked in a funny way but I think it is still easy enough to answer. Since moving clocks run slow, we have:

$$t_0 = \frac{1}{\gamma} t = \frac{1}{4} t \Rightarrow \gamma = 4 \Rightarrow 16 = \frac{1}{1-\beta^2} \Rightarrow 1-\beta^2 = \frac{15}{16} \Rightarrow \beta = 0.968$$

How about twice the normal lifespan? The answer is $\beta = 0.866$.

(3) Approximately how fast would a person have to go in order to visit a galaxy which is 200,000 ly distant within his lifespan?

Revised for 2018

We need to express the distance in a more understandable way:

$$D = 2 \times 10^5 \times c \times [1\,yr]$$

I recommend against using the approximation here $1\,yr = \pi \times 10^7 s$.

According to the fixed frame, the moving frame is seen to move with a velocity v. So according to this fixed frame:

$$D = 2 \times 10^5 \times c \times [1\,yr] = v\,t$$

where the time is in the fixed from. We do not know this time but we do know that in the moving frame, the time is 80 years. So:

$$D = 2 \times 10^5 \times c \times [1\,yr] = v\,\gamma\,t_0 = \gamma\,v \times 80\,[yr]\,\gamma\,v \Rightarrow \frac{2 \times 10^5}{80} = \gamma\beta = 2500$$

This is solvable for the required velocity:

$$\frac{\beta^2}{1-\beta^2} = (2.5 \times 10^3)^2 = 6.25 \times 10^6 \equiv x$$

$$\beta^2 = x - x\beta^2 \Rightarrow \beta^2(1+x) = x \Rightarrow \beta = \frac{\sqrt{x}}{\sqrt{(1+x)}} = \frac{2500}{\sqrt{1+2500^2}} = .99999992$$

(4) Suppose a meter stick is tilted at an angle of 45^0 relative to the x-axis in a fixed reference frame. How fast would a moving reference frame need to be moving so that the angle was 60^0 and what would be the length of the meter stick in this reference frame?

Clarified for 2018

In Frank’s frame, the meter stick is tilted at 45^0. Since the meter stick is 1 meter long, we have $L_x = L\cos(\theta) = .707 : L_y = L\sin(\theta) = .707$. Now in Mary’s frame, L_y remains the same, namely 0.707 m. However also L_x in Mary’s frame is seen to be shorter. It is short enough in this frame so that the angle of the meter stick is seen to be 60^0. How much shorter is it?

$$L_y = L_y^{'} = L'\sin(60) = .707 \Rightarrow L' = \frac{.707}{\sin(60)} = 0.816\,\mathrm{m}.$$

$$L_x^{'} = 0.816\cos(60) = 0.408\,\mathrm{m}$$

So in the moving frame the length of the meter stick is 0.816 m

Now the length in Frank’s frame is the proper length (since the meter stick is at rest with respect to Frank). This means that

$$L_x^{'} = \frac{1}{\gamma} L_{x,0} \Rightarrow \gamma = \frac{L_{x,0}}{L_x^{'}} = \frac{.707}{.408} = 1.733 = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow 3 = \frac{1}{1-\beta^2} \Rightarrow 1-\beta^2 = \frac{1}{3} \Rightarrow \beta = \sqrt{1-\frac{1}{3}}$$

$$\beta = 0.816 \Rightarrow v = 0.816c$$