A simple consequence of everyone measuring the same value for the speed of light

Consider two observers, Frank in system O and Mary in system O'. Let Mary move at some velocity $v = \beta c$ relative to Frank along the x-axis. Both Frank and Mary are holding lasers. When Mary is exactly beside Frank, their clocks both read 0 and they pulse their lasers. What do both observers see after 1 sec?

According to Frank, both laser pulses have traveled $3x10^8$ m in that one second: both pulses are $3x10^8$ m from Frank's position.

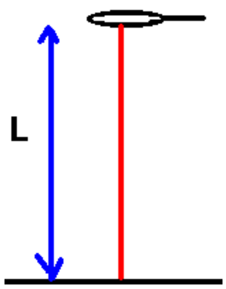
Mary, during that one second, traveled a distance of $x_m = 1\sec \times \beta c$ in that same time. However, since all observers see the same speed of light, Mary reports that both pulses are $3x10^8$ m away.

But remember, Mary has moved the distance x_m that Frank has not moved through.

How can both observers possibly report different results?

The answer can only be that a time interval of 1s in Mary's frame can not be the same as a time interval of 1s in Frank's frame. According to Frank, Mary's clock must be running slow. But, Mary says that Frank's meter stick is too short. This is the downfall of simultaneity: two events which are simultaneous in one reference frame are not necessarily simultaneous in another reference frame.

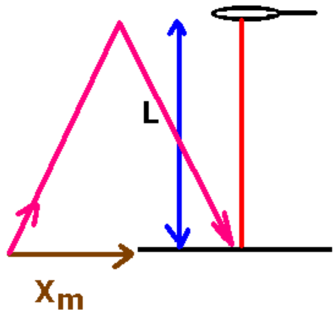
Suppose Frank and Mary have mirrors floating over their heads. This is kind-of a nice mirror in that it matches the motion of their particular coordinate system. The mirror is always over their heads independently of what the motion of their coordinate system is (so long as it is not being accelerated).



The time it takes a pulse to travel through this distance and back is $t_0 = \frac{2L}{c}$. This is completely independent of if the mirror is moving or not. We will call this the proper time: it does not care about movement or not. An observer in this frame of reference will measure this time.

Proper time t_0 is the time interval between two events measured by an observer who is at rest relative to the two events and sees the events occur at the same point in space.

Now let the frame of reference move (with Mary) at a velocity of $v = \beta c$ along the x-direction (perpendicular to the direction of the laser pulse). We need to determine what Frank observes. Ultimately, Frank will see this:



The additional distance can only be accounted for if somehow Mary's clock ran slower than Frank's clock. But now, we can calculate this exactly. Then, using the time dilation, we can return to the previous problem to obtain the length contraction.

Frank measures the distance that the light traveled to be: $2ct$. In this time, Mary's frame moved through a distance $2\beta ct$. Mary measured the pulse to travel through a distance of $2ct_0$. These three quantities are related by Pythagorean's theorem:

$$c^2t^2 = \beta^2c^2t^2 + c^2t_0^2$$

We want to solve this for t. The result is:

$$c^2\left[1-\beta^2\right]t^2 = c^2t_0^2 \Rightarrow t = \frac{1}{\sqrt{1-\beta^2}}t_0 \Rightarrow t = \gamma t_0$$

That is the required time dilation factor.

Now the factor γ is always greater than or equal to 1. This means that for every tick of Mary's clock, Frank's clock ticks more. In other words, Mary's clock runs slow.

Now let's try to make a length measurement. I want to define proper length first.

Proper length L_0 is the distance between two points in a frame of reference which is at rest with respect to those two points.

How to make a length measurement.

Let Mary be walking with a constant speed $v = \beta c$ parallel to a stick and she is carrying a pointer. Frank is in the reference frame of the stick. Mary measures the length of the stick by determining how long it takes for her to walk past the stick, according to a stopwatch in her moving frame of reference. She does this by noticing when her pointer is at one end of the stick (thus starting her stopwatch) and then noticing when her pointer is at the other end of the stick (thus stopping her stopwatch). This distance is given by $L_m = \beta ct_m$. Frank also measures the length of the stick by noting when Mary's pointer is at each end of the stick. Frank measures the following result: $L_F = \beta ct_F$.

Now we need to determine which of these times is the proper time in order to relate the two measurements. We need to look at how proper time is defined to determine this.

Proper time t_0 is the time interval between two events measured by an observer who is at rest relative to the two events and sees the events occur at the same point in space.

Now it is Mary who is looking at the same pointer in her reference frame. Frank has to rotate his head to follow this so clearly Frank's time can not be the proper time. By process of elimination, it must be Mary's time which is therefore the proper time. Don't worry about the fact that Frank sees Mary's frame moving, Mary sees her frame at rest and she sees the defining events at the same location in space. This then gives us the connection between Frank's time increment and Mary's time increment:

$$t_f = t : t_m = t_0 \Rightarrow t_f = \gamma t_0$$

We can now relate the two lengths:

$$L_m = \beta c t_0$$

$$L_F = \beta c t_F = \beta c \gamma t_0$$

Now let's determine which of these lengths is the proper length. Again, looking at the definition we have:

Proper length L_0 *is the distance between two points in a frame of reference which is at rest with respect to those two points.*

Clearly the distance in Frank's frame of reference is the proper length because it is his reference frame which is at rest with respect to the stick. This means Frank measures L_0.

So this is then the way the measurements are related:

$$L_m = \beta c t_0$$

$$L_0 = \beta c \gamma t_0$$

We divide the two to obtain:

$$\frac{L_m}{L_0} = \frac{1}{\gamma} \Rightarrow L_m = \frac{1}{\gamma} L_0$$

I think we can now lose the "m" subscript. We can simply say then:

$$L = \frac{1}{\gamma} L_0$$

This contracted length is always shorter than or equal to the proper length.
Remember the time dilation is given by:

$$t = \gamma t_0$$

which is always greater than or equal to the proper time.

I guess you can remember:
moving clocks run slow and moving meter sticks expand.

There is an important warning: If two events are separated by both time and space (meaning they don't occur at the same location in space), they can not be related simply by multiplying of dividing by the gamma factor.

Here are some simple problems to get started on:

(1) Frank measures a time interval of 5s on his watch during which time Mary's spaceship passes which is moving at a speed of 0.9999c. What time increment does Mary measure?

I believe the way to best answer this question is to remember moving clocks run slow. But ultimately it is $t = \gamma t_0 \Rightarrow t_0 = \frac{1}{\gamma} t = 0.01414(5) = 0.07s$

You can look at this problem another way if you like. Suppose an exotic particle lives for 0.07 s in its own rest frame. If this exotic particle travels at 0.9999c, relative to another frame, how long does the particle live? The answer is 5s.

(2) How fast does a rocket need to go so that the occupant of the ship lives 4 times the normal lifespan of a fixed observer?

This is asked in a funny way but I think it is still easy enough to answer. Since moving clocks run slow, we have:

$$t_0 = \frac{1}{\gamma} t = \frac{1}{4} t \Rightarrow \gamma = 4 \Rightarrow 16 = \frac{1}{1-\beta^2} \Rightarrow 1-\beta^2 = \frac{1}{16} \Rightarrow \beta^2 = \frac{15}{16} \Rightarrow \beta = 0.968$$

How about twice the normal lifespan? The answer is $\beta = 0.866$

(3) Approximately how fast would a person have to go in order to visit a galaxy which is 200,000 ly distant within his lifespan?

Let's assume a person lives to be 80 years old. Then the time taken according to a fixed reference frame is:

$t = \frac{200000}{80} = 2500\text{yr}$

This gives us the gamma factor:

$$2500 = \gamma(80) \Rightarrow \gamma = 31.25 = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow 977 = \frac{1}{1-\beta^2} \Rightarrow 1-\beta^2 = \frac{1}{977} \Rightarrow \beta = .9995$$

How fast would he need to go to get there and back in a lifetime?

$$5000 = \gamma(80) \Rightarrow \gamma = 62.5 = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow 3906.25 = \frac{1}{1-\beta^2} \Rightarrow 1-\beta^2 = \frac{1}{3906.25} \Rightarrow \beta = .9999$$

(4) Suppose a meter stick is tilted at an angle of 45^0 relative to the x-axis in a fixed reference frame. How fast would a moving reference frame need to be moving so that the angle was 60^0 and what would be the length of the meter stick in this reference frame?

This is a bit harder. However, it is only lengths that are parallel to the direction of velocity that are affected. Those perpendicular are not affected. This means, then, that the x-direction gets shorter while the y-direction is unchanged. How much shorter?

$$L_x = \cos(60) = 0.5\text{m} : L_y = \sin(45) = .707$$

The proper length here is the original x-projection which is given by:

$$L_{x,0} = \cos(45) = .707\text{m}$$

The required gamma factor is then:

$$L = \frac{1}{\gamma} L_0 \Rightarrow \gamma = \frac{L_0}{L} = \frac{.707}{.5} = 1.4142 = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow 1.99996 = \frac{1}{1-\beta^2} \Rightarrow \beta^2 = 0.5 \Rightarrow \beta = .707$$

In the moving frame, the meter stick has a length given by:

$$L = \frac{L_y}{\sin(60)} = \frac{.707}{.866} = 0.816\text{m}$$