Blackbody radiation

Observations:

(1) Maximum wavelength of radiation distribution shifts to smaller wavelengths as temperature in increased: Wien's displacement law:

$$\lambda_{max} T = 2.898 \times 10^{-3}\,\mathrm{mK}$$

(2) Total power radiated increases with temperature.

Intensity:

Remember from general physics how intensity is defined:

$$\text{Intensity} \equiv \frac{\langle \text{power} \rangle}{\text{Area}}$$

which is **normal** to a detector area.

Stefan-Boltzman law

It is possible to derive the Stefan-Boltzman law which states

$$R(T) = \varepsilon A \sigma T^4 : \varepsilon = \text{emissivity} : \sigma = 5.6705 \times 10^{-8}\,\mathrm{W/(m^2 K^4)}$$

meaning: the total power radiated from a black body of surface area A is proportional to the 4th power of the absolute temperature in Kelvin. However, it pre-dated Planck's law. In fact, Planck's law is what allows us to obtain the Stefan-Boltzman law. It is done in the following way: According to Planck's law the time average power/(unit area Unit solid angle) radiated at a temperature T and at a frequency f will be:

$$I(f,T) = \frac{2hf^3}{c^2} \frac{1}{e^{\frac{hf}{kT}} - 1}$$

where h is Planck's constant. We can find the total power radiated at a temperature T by integrating over all frequencies and all solid angle. The solid angle for a sphere is defined by:

$$\Omega = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\varphi=0}^{\varphi=2\pi} \sin\theta\, d\theta\, d\varphi$$

However, in order to find the power passing though the detector we need to take the dot product of the dot of the radiation direction with the normal to the detector area which involves a factor of $\cos(\theta)$. This is because of that pesky requirement that in order to detect light, it needs to pass through the detector.

We are now ready to set up the required integral for the total Intensity detected at a particular temperature T:

$$R(T) = \frac{2h}{c^2} \int_{f=0}^{f=\infty} \left[\frac{f^3}{e^{\frac{hf}{kT}} - 1} \right] df \int_{\varphi=0}^{\varphi=2\pi} d\varphi \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin(\theta)\cos(\theta)\, d\theta$$

The angular dependence integrates to become π.

This then gives the rest of the integral

$$R(T) = \frac{2h}{c^2}\pi \int_{f=0}^{f=\infty} \left[\frac{f^3}{e^{\frac{hf}{kT}} - 1} \right] df$$

To do the integral, let $x \equiv \frac{hf}{kT}$

Then, the function appears as:

$$R(T) = \frac{2h}{c^2}\pi \left(\frac{kT}{h}\right)^3 \left(\frac{kT}{h}\right) \int_{x=0}^{x=\infty} \left[\frac{x^3}{e^x - 1} \right] dx = \frac{2\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_{x=0}^{x=\infty} \left[\frac{x^3}{e^x - 1} \right] dx$$

The integral

$$\int_{x=0}^{x=\infty} \left[\frac{x^3}{e^x - 1} \right] dx$$

is an example of the Bose-Einstein integral which Planck did solve. In this case the result is

$$\int_{x=0}^{x=\infty} \left[\frac{x^3}{e^x - 1} \right] dx = \frac{\pi^4}{15}$$

which is not going to be particularly easy to obtain.

Evaluation of Planck's Integral

Here is one way that you can find (include corrections as needed):

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}} = e^{-x}\left[1 + e^{-x} + e^{-2x} + \ldots\right] = e^{-x} \sum_{m=0}^{m=\infty} e^{-mx}$$

The integral then appears as

$$\int_{x=0}^{x=\infty} x^3 e^{-x} \sum_{m=0}^{m=\infty} e^{-mx} dx = \sum_{m=0}^{m=\infty} \int_{x=0}^{x=\infty} x^3 e^{-(m+1)x} dx$$

Now let $n \equiv m + 1 : m = 0 \Rightarrow n = 1$

The integral is then

$$\sum_{n=1}^{n=\infty} \int_{x=0}^{x=\infty} x^3 e^{-nx} dx$$

This can be done easily to obtain (in a modern age, use the integrator):

$$\sum_{n=1}^{n=\infty} \int_{x=0}^{x=\infty} x^3 e^{-nx} dx = -\sum_{n=1}^{n=\infty} \left[\frac{e^{-nx}}{n^4} \left(n^3 x^3 + 3n^2 x^2 + 6nx + 6 \right) \right] \Bigg|_{x=0}^{x=\infty}$$

Now the exponential always goes to zero faster than any positive power of x at infinity. We are thus left with the result:

$$\sum_{n=1}^{n=\infty} \int_{x=0}^{x=\infty} x^3 e^{-nx} dx = \sum_{n=1}^{\infty} \frac{6}{n^4}$$

This is a special case of the Riemann Zeta function which is defined by:

$$\zeta(s) \equiv \sum_{n=1}^{n=\infty} \frac{1}{s^n}; re(s) > 1 .$$

These have been calculated and you can even find online zeta function calculators. The result for s=4 (our case) is

$$\zeta(4) = \frac{\pi^4}{90}$$

So our integral becomes:

$$\sum_{n=1}^{n=\infty} \int_{x=0}^{x=\infty} x^3 e^{-nx} dx = 6\zeta(4) = \frac{\pi^4}{15}$$

Thus the time average power radiated normal to a surface area A per unit area is: (noting that I switched detector with the radiating area: this is ok):

$$R(T) = \frac{2\pi h}{c^3}\left(\frac{kT}{h}\right)^4 \frac{\pi^4}{15}$$

The power radiated by the area A is then given by:

$$\langle power \rangle = A\frac{2\pi h}{c^3}\left(\frac{k}{h}\right)^4 \frac{\pi^4}{15} T^4 = \sigma A T^4 ; \sigma = \frac{2\pi^5 k^4}{15 c^3 h^3}$$

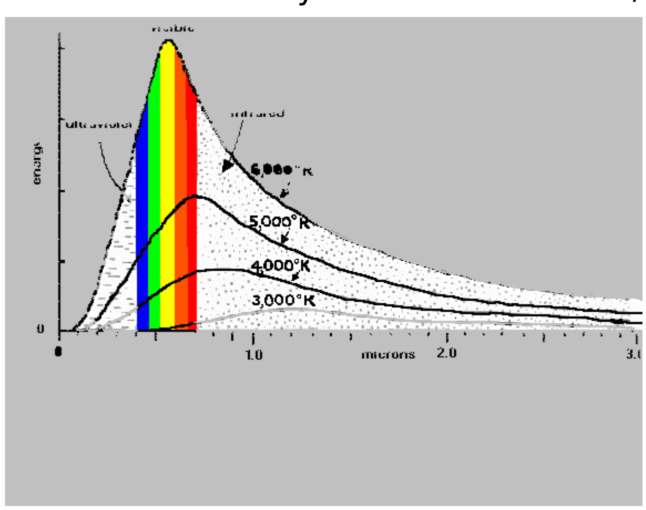
This radiated power is actually called the irradiance. For a perfect blackbody this is the case. However for a body which is not necessarily a black body, but is characterized by the emissivity ε:

$$\langle power \rangle = \varepsilon \sigma A T^4 .$$

For blackbody radiation, ε=1.

The fact that this power converged, was Planck's triumph (in addition to obtaining the theoretical value for σ). You can see if we used the Wien displacement law, the integral would diverge which is known as the ultraviolet catastrophe.

$$R(T) = \varepsilon\sigma T^4 : \varepsilon = \text{emissivity} : \sigma = 5.6705 \times 10^{-8} W/(m^2K^4)$$



Rayleigh-Jeans formula from classical EM with continuous energy:

$$I(\lambda, T) = \frac{2\pi ckT}{\lambda^4} : k = \text{Boltzmann Constant}$$

Ultraviolet catastrophe solved by Planck assuming quantized oscillators:

$$I(\lambda, T) = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} : h = \text{Planck's constant}$$